Sketching for Large-Scale Learning of Mixture Models

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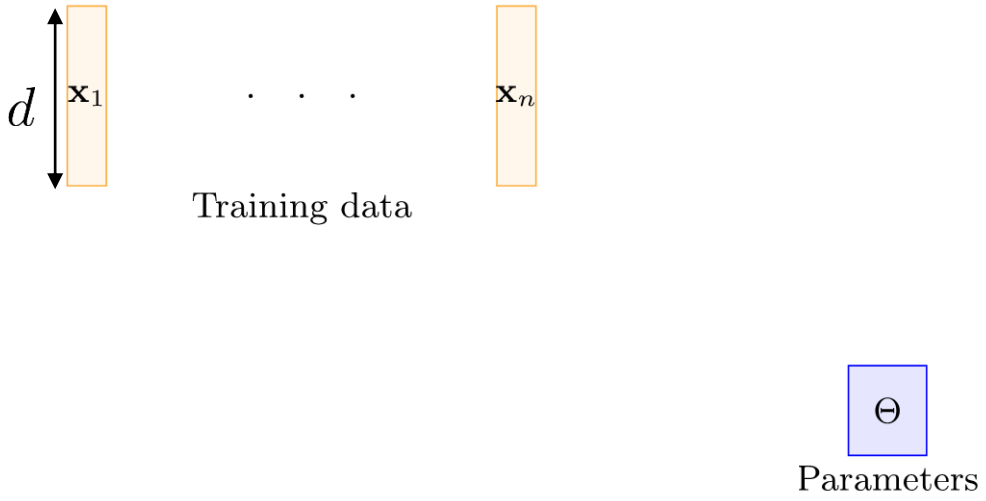
Adv. Rémi Gribonval

Outline

Paths to Compressive Learning

Goal : Compute parameters Θ from a **large** database.



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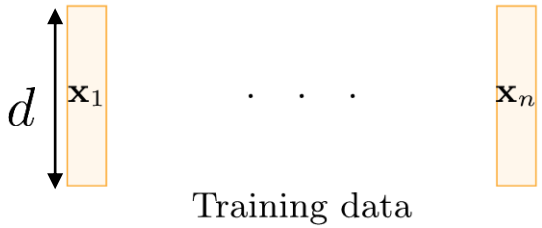

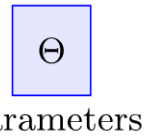

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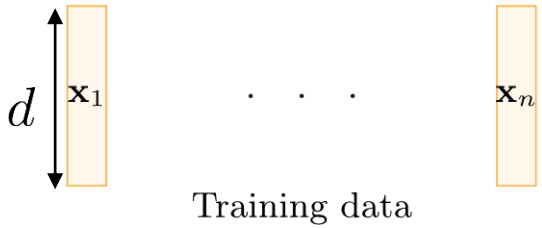

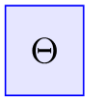

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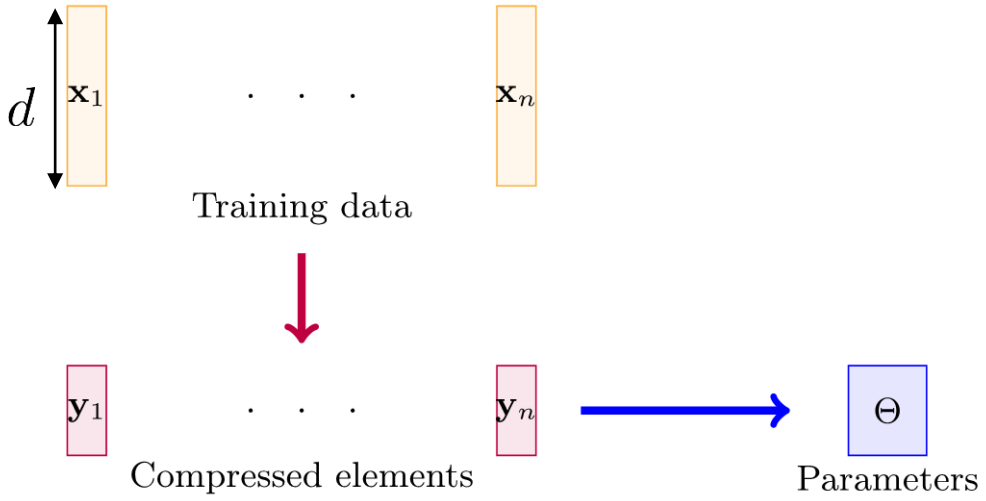

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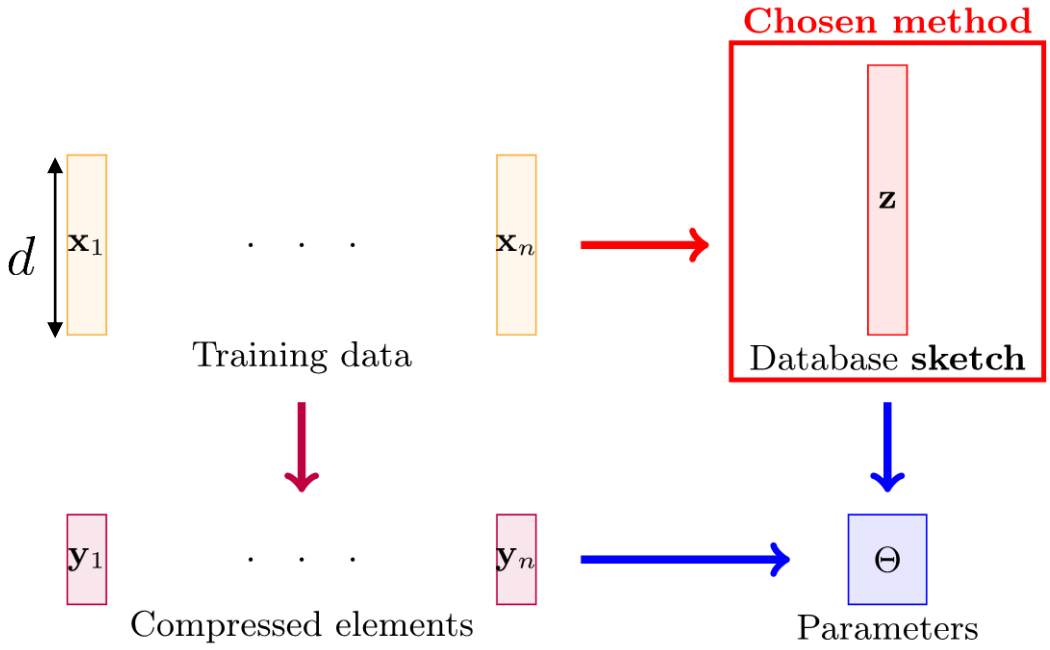
- Large n (fat) **« Big data »**
  - See e.g. [Cormode 2011]



Inspiration

- **Sketch…**

- Contains particular info about the database
- Maintained **online**

[Cormode 2011]

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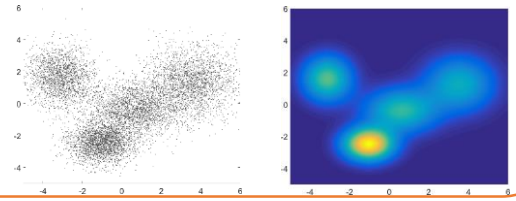
  - Knowledge about **underlying probability distribution**



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  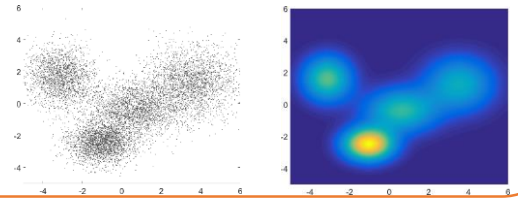
- **…by Compressive Sensing**

  Recover « low-dimensional » object from few linear measurements (ex : sparse vector, low-rank matrix…)

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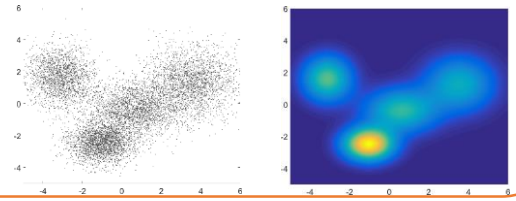
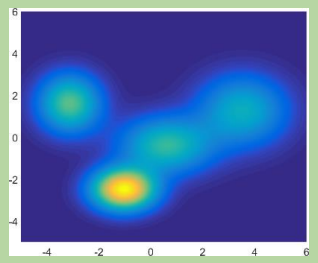


- **…by Compressive Sensing**

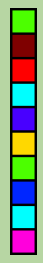
Recover « low-dimensional » object from few linear measurements (ex : sparse vector, low-rank matrix…)

Sketch = measurements of underlying probability distribution



$p \in \mathcal{P}$

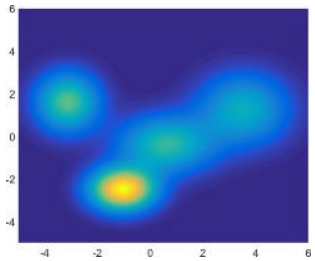
$\mathcal{A}$



$\mathbf{z} = \mathcal{A}p \in \mathbb{C}^m$

Generalized Moments

(Generalized) Compressive Sensing



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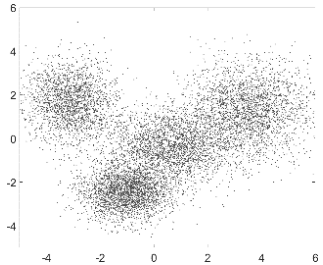
Practical Compressive Learning



$\hat{p} = (\sum_i \delta_{\mathbf{x}_i})/n$

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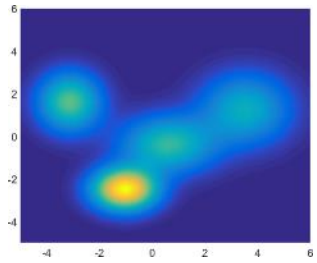

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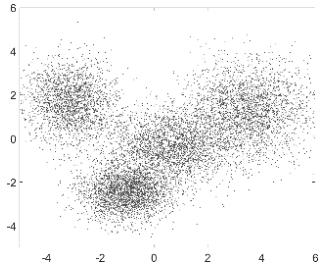

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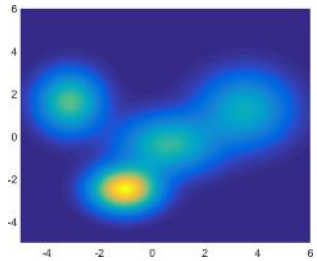

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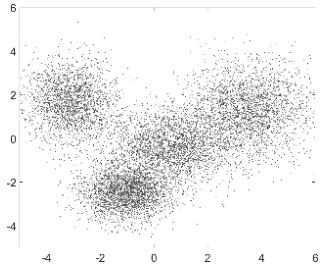

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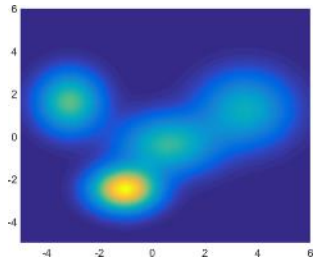

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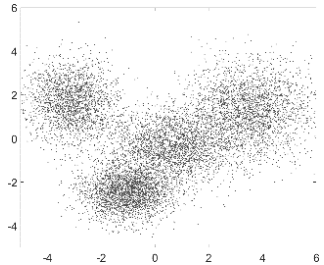

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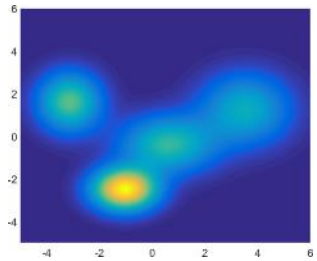

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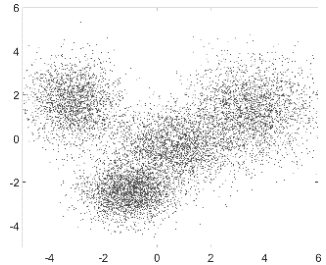

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Robustness of learning Alg. ?

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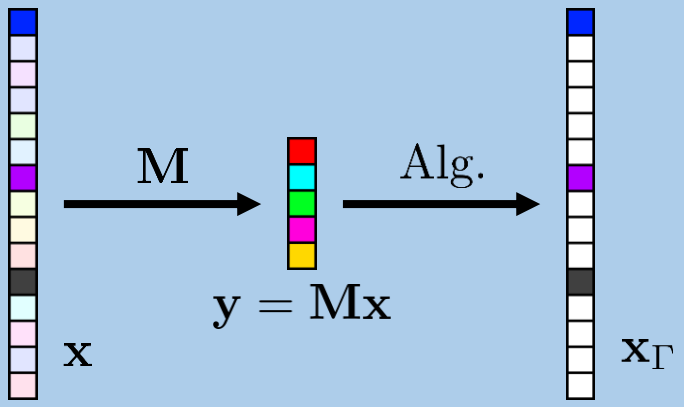
Mixture Model Estimation

Usual Compressive Sensing



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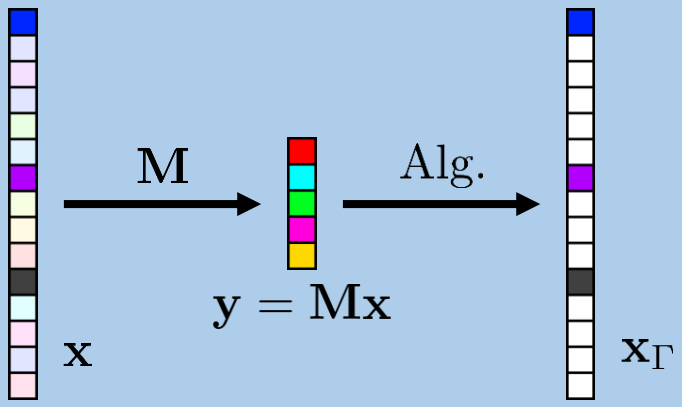

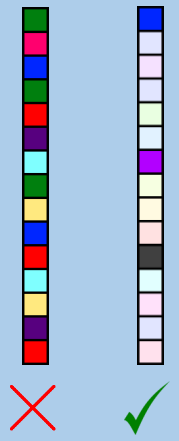

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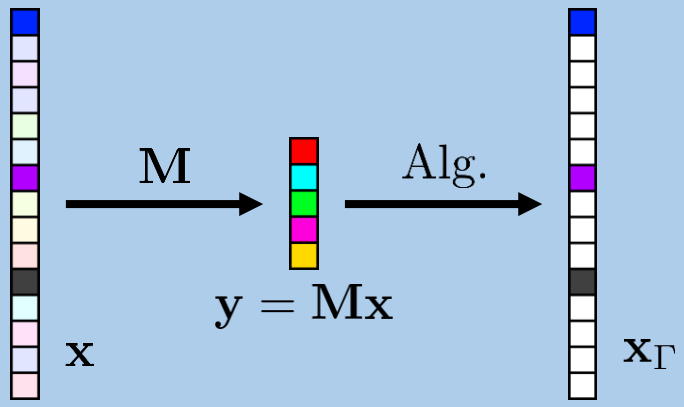

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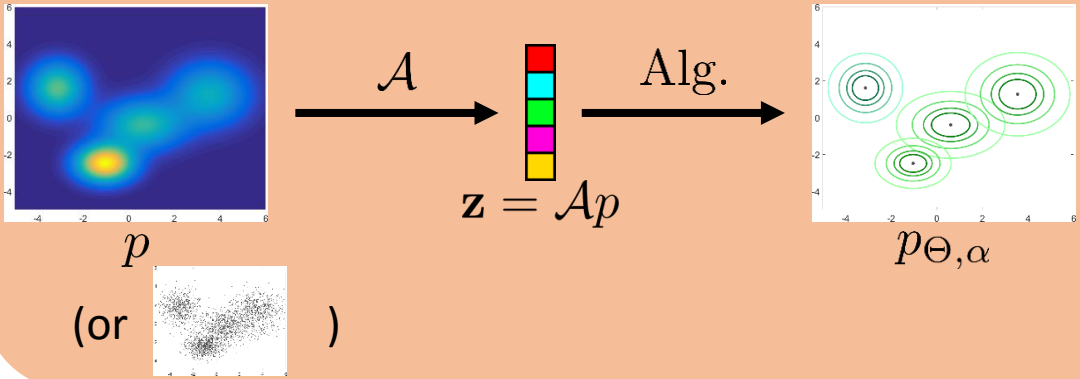

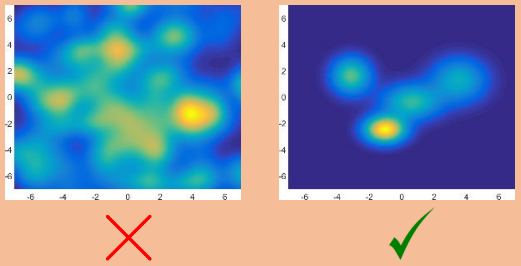

$$p \approx p_{\Theta,\alpha} = \sum_{l=1}^{k} \alpha_l p_{\theta_l}$$

Infinite, continuous dictionary !

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  - Greedy algorithm inspired by Compressive Sensing
  - Application to K-means, GMM with diagonal covariance

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- Theoretical Analysis (Section 4)
  - Information-preservation guarantee
  - Infinite-dimensional Compressive Sensing

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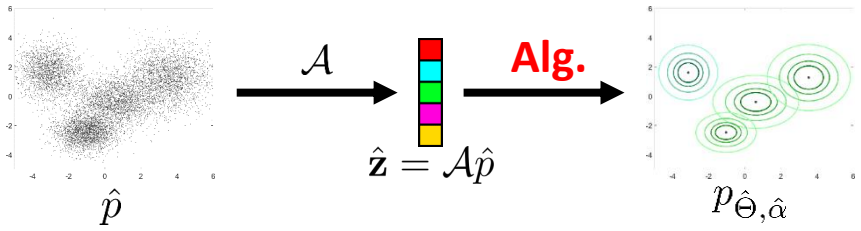
3. Results
4. Theoretical analysis
5. Conclusion and outlooks

Cost function



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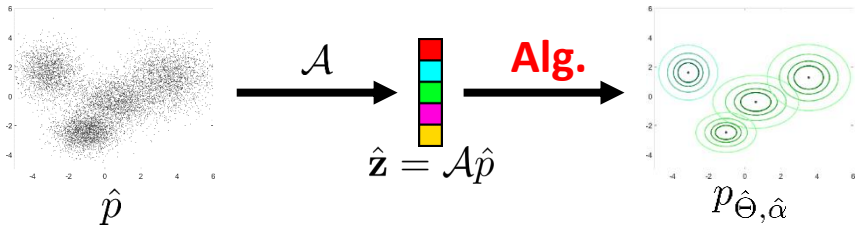

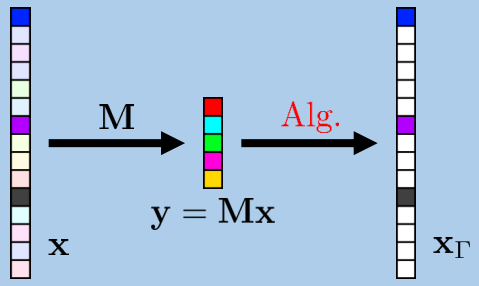

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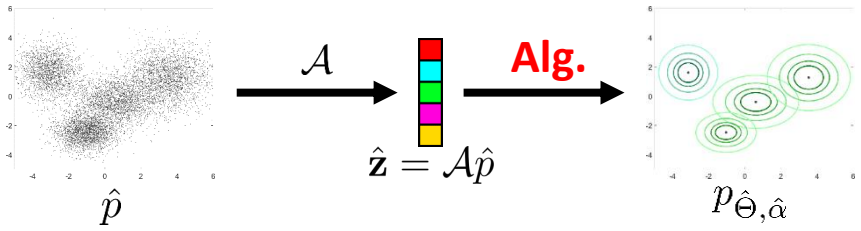

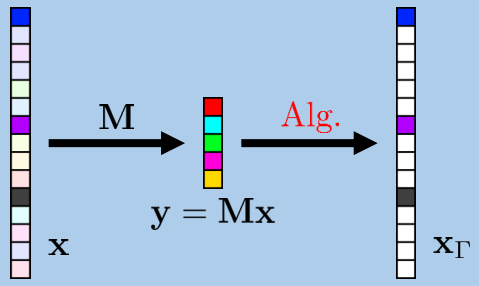

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- « Ideal » decoding scheme

See [Foucart 2013]

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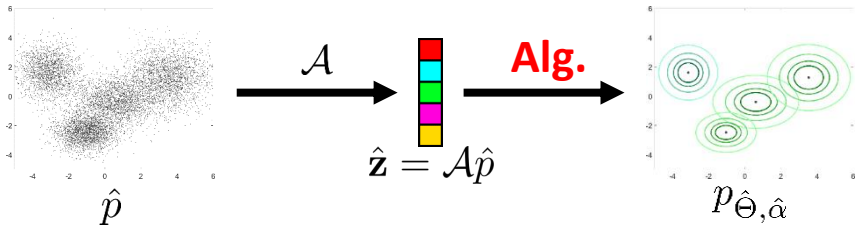

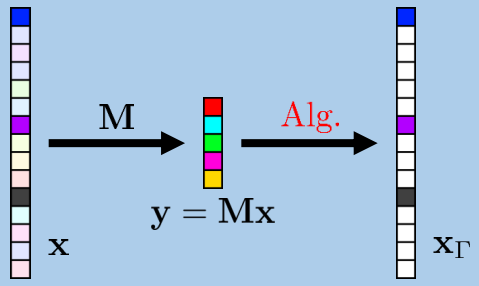

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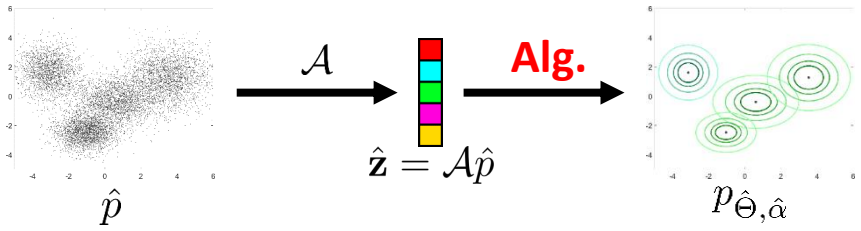

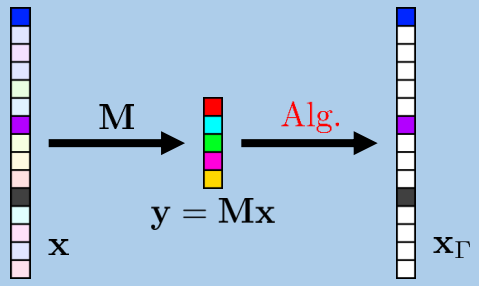

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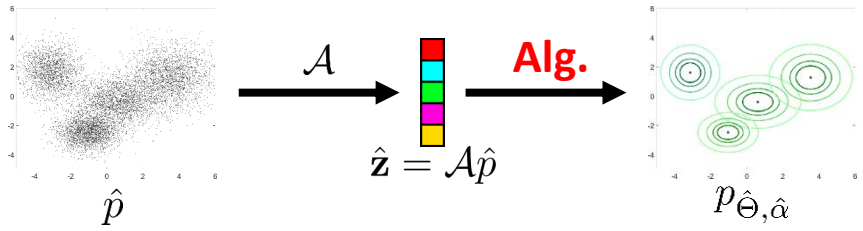

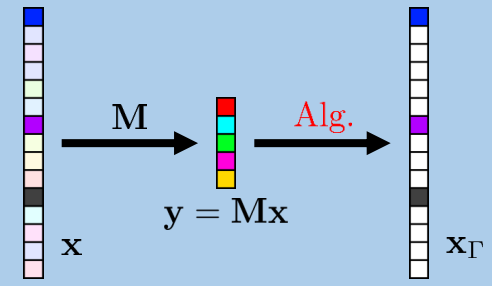

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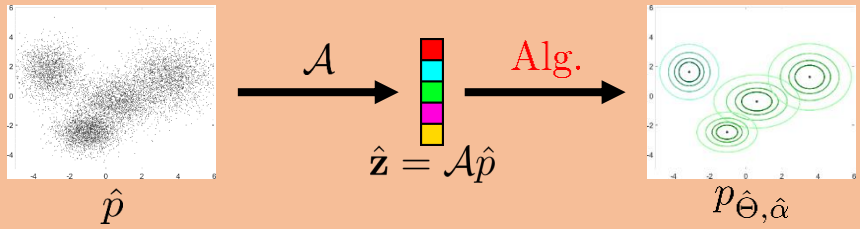

$$\min_{\Theta,\alpha} \|\hat{\mathbf{z}} - \mathcal{A}p_{\Theta,\alpha}\|_2$$

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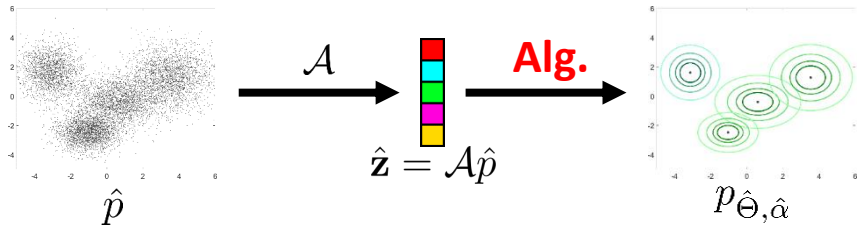
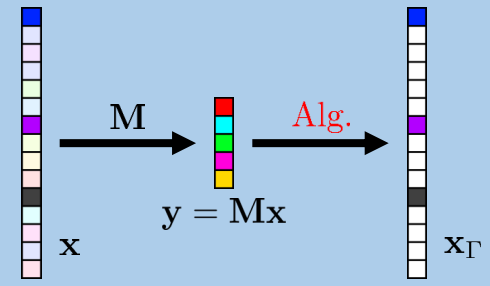
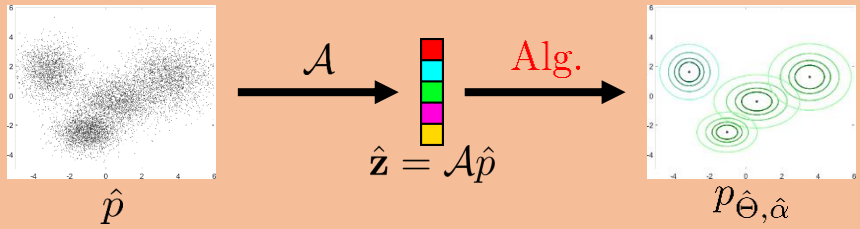




$$\min_{\|\mathbf{x}\|_0 \leq s} \|\mathbf{y} - \mathbf{M}\mathbf{x}\|_2$$

- « Ideal » decoding scheme
- NP-complete
- Two approaches:
  - Convex relaxation
  - Greedy

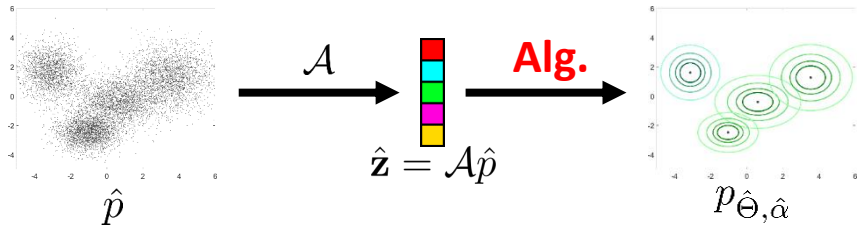
See [Foucart 2013]

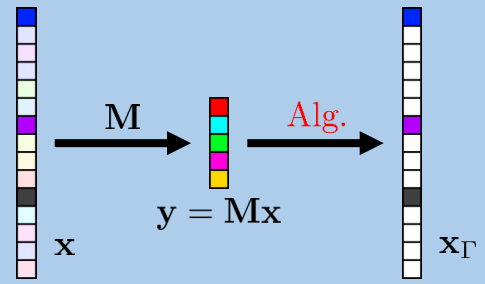


$$\min_{\Theta,\alpha} \|\hat{\mathbf{z}} - \mathcal{A}p_{\Theta,\alpha}\|_2$$

- Ideal decoding scheme ✓ (Section 4)

Cost function





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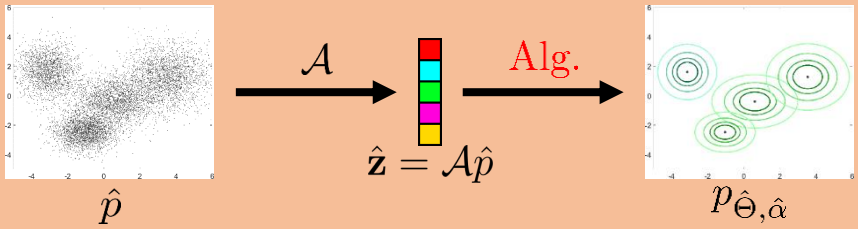

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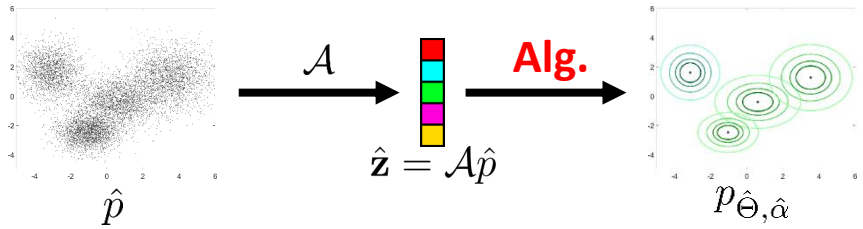
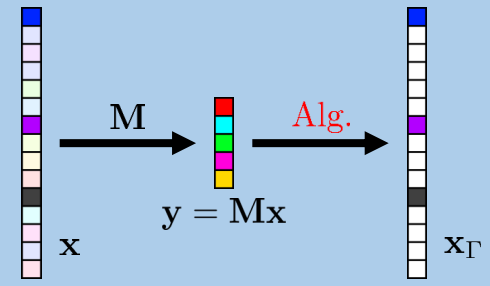
- Ideal decoding scheme ✓ (Section 4)
- Highly non-convex

Cost function





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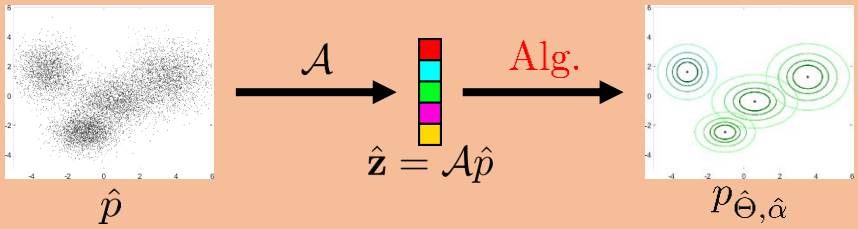

$$\min_{\Theta,\alpha} \|\hat{\mathbf{z}} - \mathcal{A}p_{\Theta,\alpha}\|_2$$

- Ideal decoding scheme ✓ (Section 4)
- Highly non-convex
- Two approaches:
  - Convex relaxation *[Bunea 2010]* ✗
  - Greedy **Proposed**

Proposed algorithm *(Keriven, Bourrier, Gribonval, Pérèz)*

Orthogonal Matching Pursuit (OMP)
[Mallat 1993, Pati 1993]

1. Add atom most correlated to residual
2. Perform Least-Squares
3. Repeat until desired sparsity

Proposed algorithm *(Keriven, Bourrier, Gribonval, Pérèz)*

OMP with Replacement (OMPR)
[Jain 2011]

1. Add atom most correlated to residual
2. **Perform Hard-Thresholding** (if necessary)
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4. Repeat **twice desired sparsity**

Similar to CoSAMP [Needell 2008] or SubSpace Pursuit [Dai 2009]

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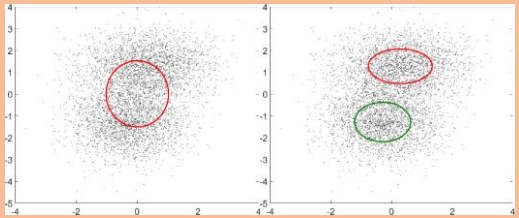
Similar to CoSAMP [Needell 2008] or SubSpace Pursuit [Dai 2009]

Compressive Learning OMPR (CLOMPR) (proposed)

1. Add atom most correlated to residual **with gradient descent** (local min.)
2. Perform Hard-Thresholding
3. Perform **Non-Negative** Least-Squares
4. **Perform gradient descent on all parameters, initialized with current ones** (local min.)

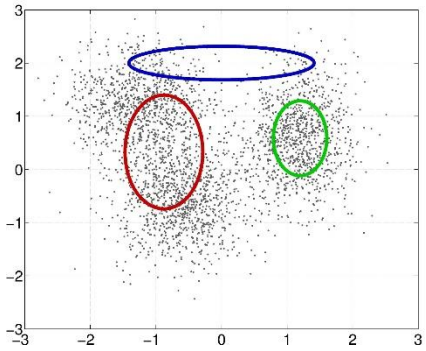


We cannot just add a component

5. Repeat twice desired sparsity

CLOMPR : illustration

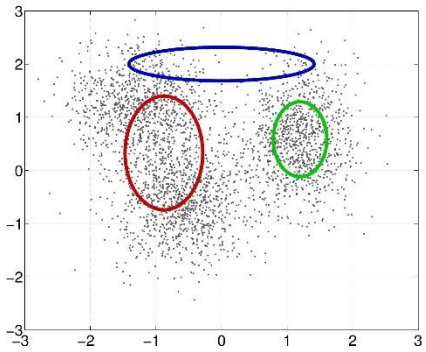
(schematic illustration of one iteration)



Goal : 3-GMM. Intermediary support.

CLOMPR : illustration

(schematic illustration of one iteration)



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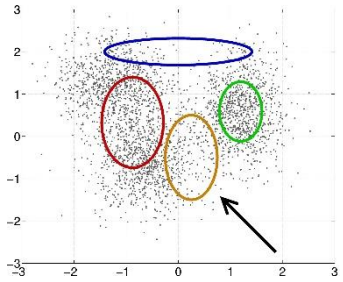

1 : Add atom **with gradient descent** (local min. of correlation)

CLOMPR : illustration

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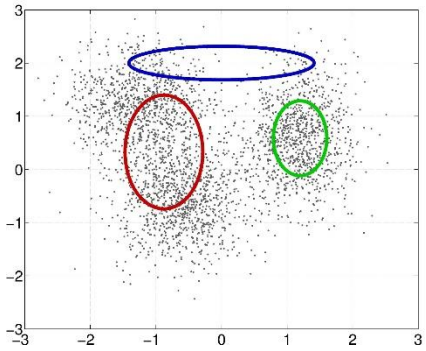

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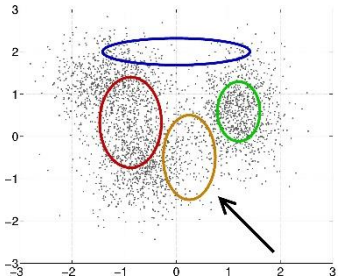

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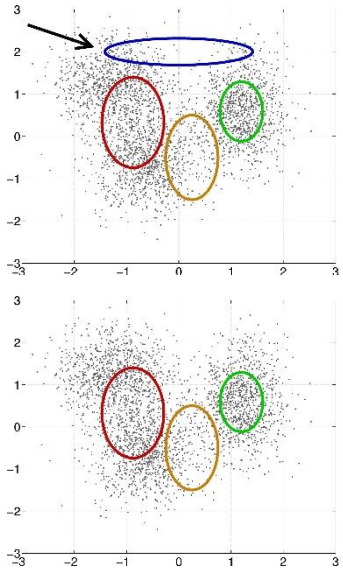

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3 : **Non-Negative** Least-Squares

CLOMPR : illustration

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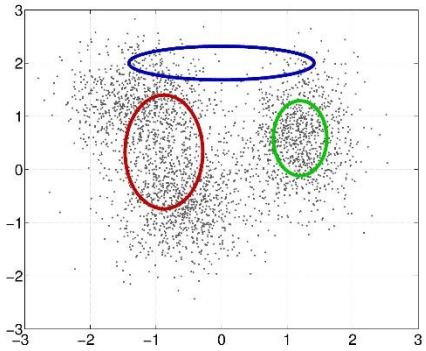

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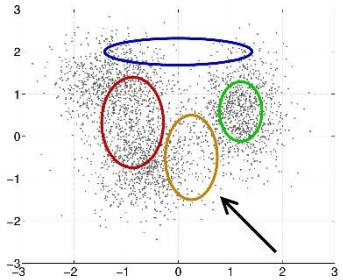

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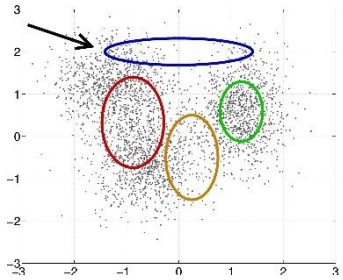

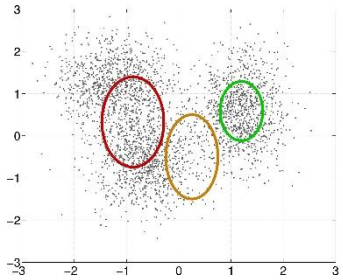

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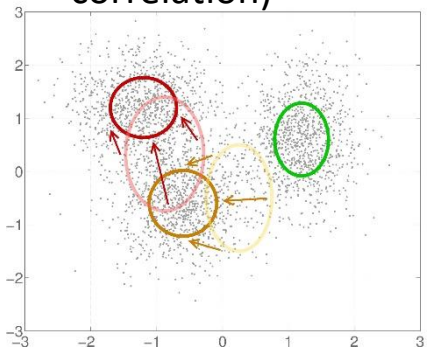

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CLOMPR : illustration

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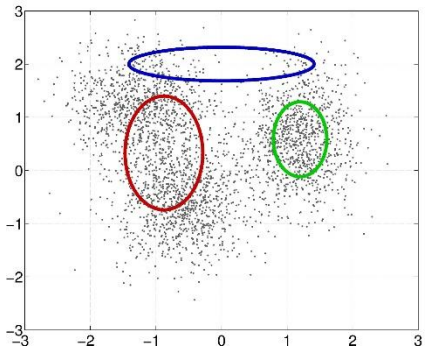
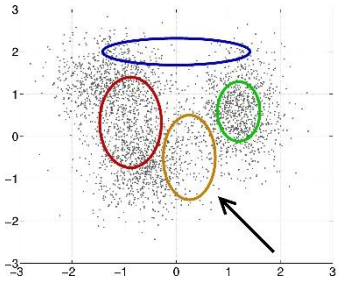


Goal : 3-GMM. Intermediary support.



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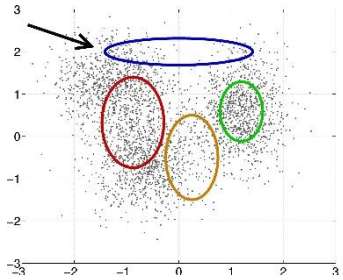

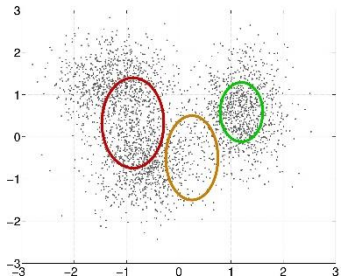

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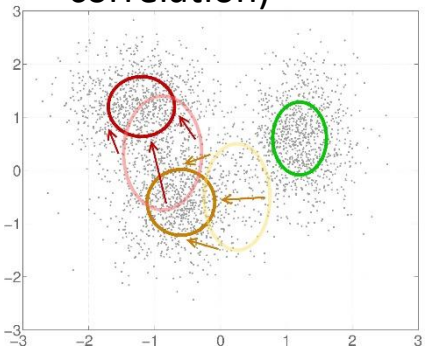

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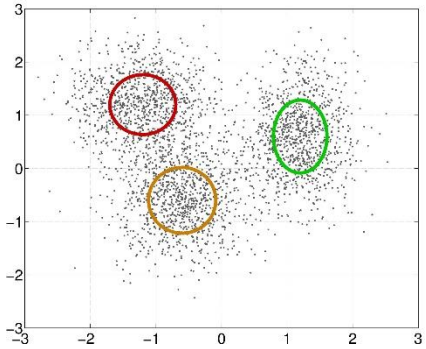

Final support

Choice of sketch

To implement CLOMPR, $\mathcal{A}p_\theta$ and $\nabla_\theta \mathcal{A}p_\theta$ must have a closed-form expression w.r.t. θ

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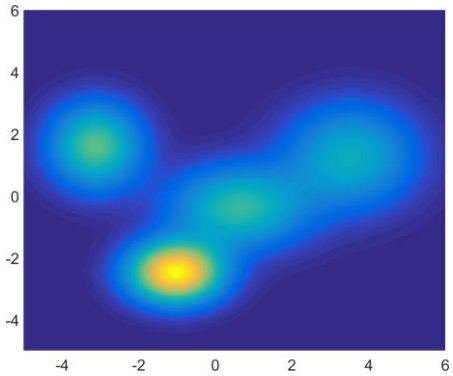
p is spatially localized



p

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To implement CLOMPR, $\mathcal{A}p_\theta$ and $\nabla_\theta \mathcal{A}p_\theta$ must have a closed-form expression w.r.t. θ

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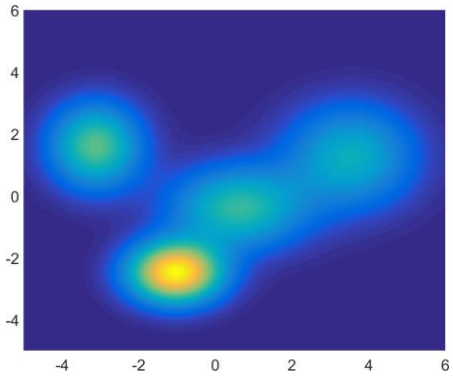
Need incoherent sampling -> Fourier sampling



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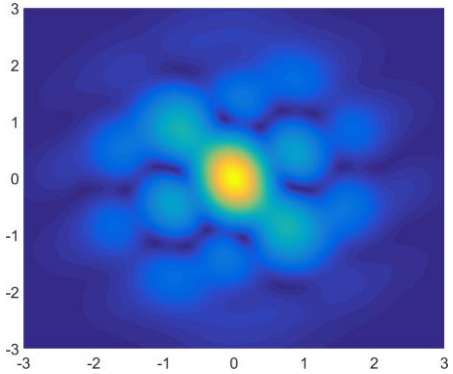
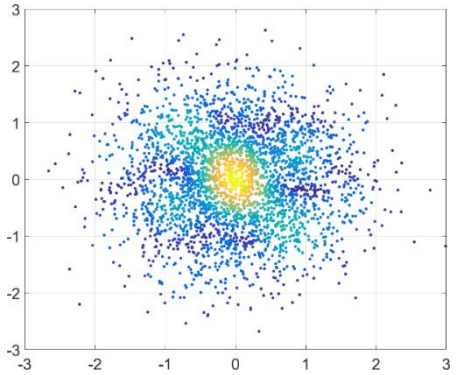


$\psi_p(\omega) = \mathbb{E}_{\mathbf{x}\sim p} e^{-i\omega^T \mathbf{x}}$



$\mathbf{z}$

Choice of sketch

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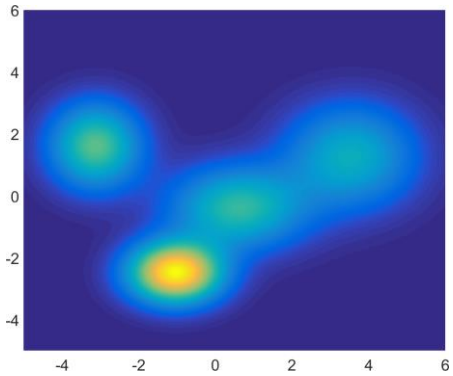

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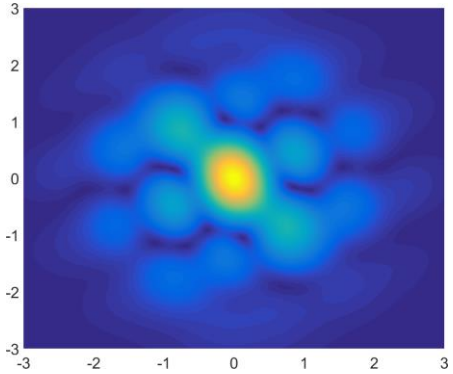

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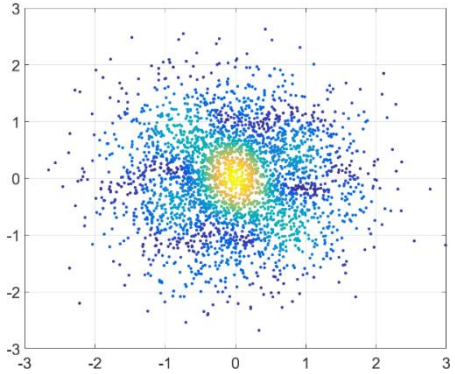

$\mathbf{z}$

$$\mathcal{A}p = \left[\psi_p(\omega_j)\right]_{j=1}^{m}$$

Closed-form for many models !
(including alpha-stable…)

Choice of sketch

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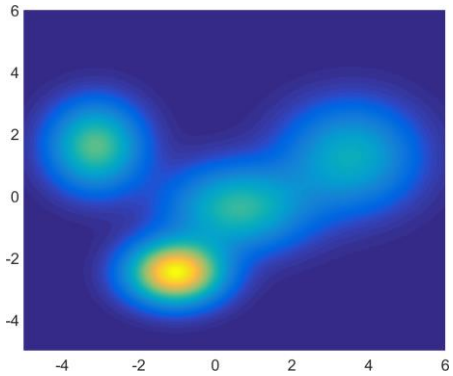

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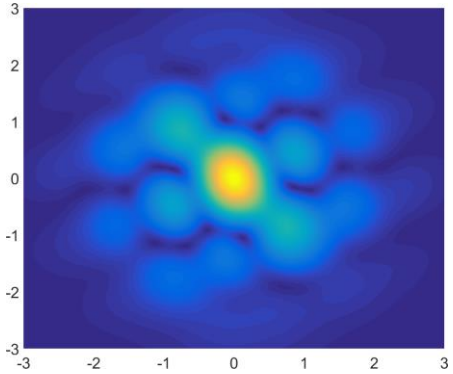

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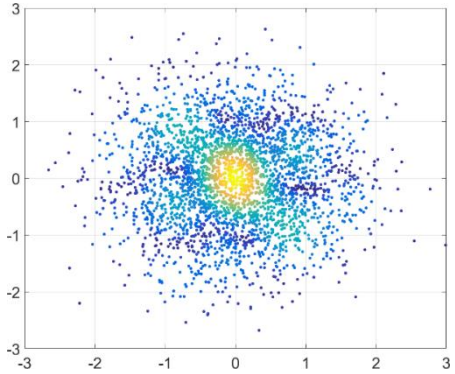

$\mathbf{z}$

$$\mathcal{A}p = \left[\psi_p(\omega_j)\right]_{j=1}^{m}$$

$$\omega_j \overset{i.i.d.}{\sim} \Lambda$$

Closed-form for many models !
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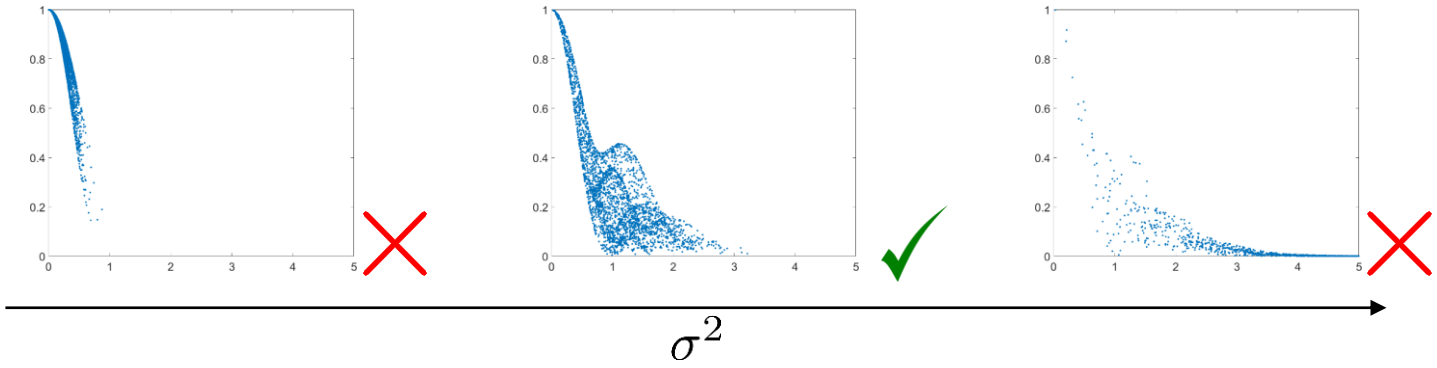
Random Fourier sampling *[Candes 2006]*
Random Fourier features *[Rahimi 2007]*

Designing the frequency distribution

Must adjust « scale » of distribution



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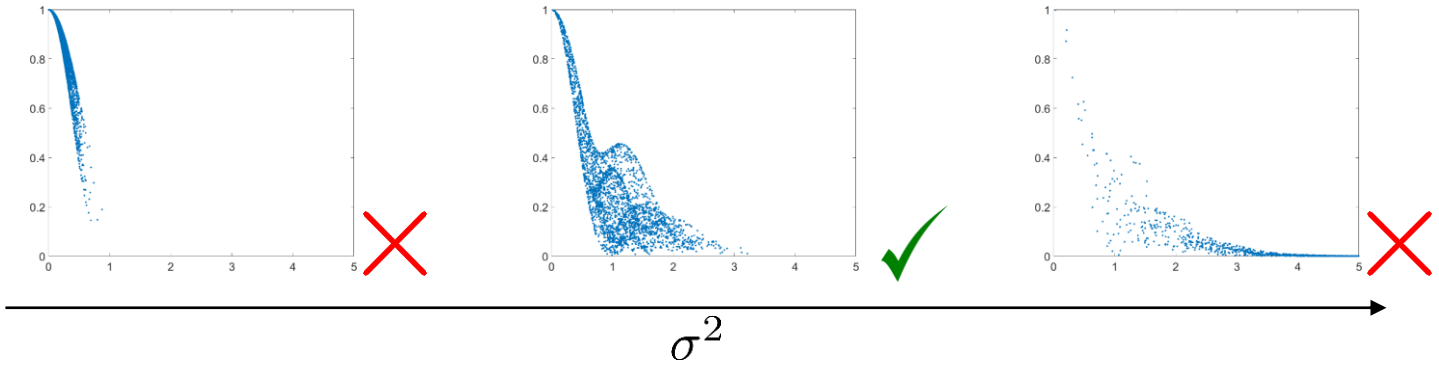

Adjust by hand

- *Not that difficult…*
- *The method is quite robust*

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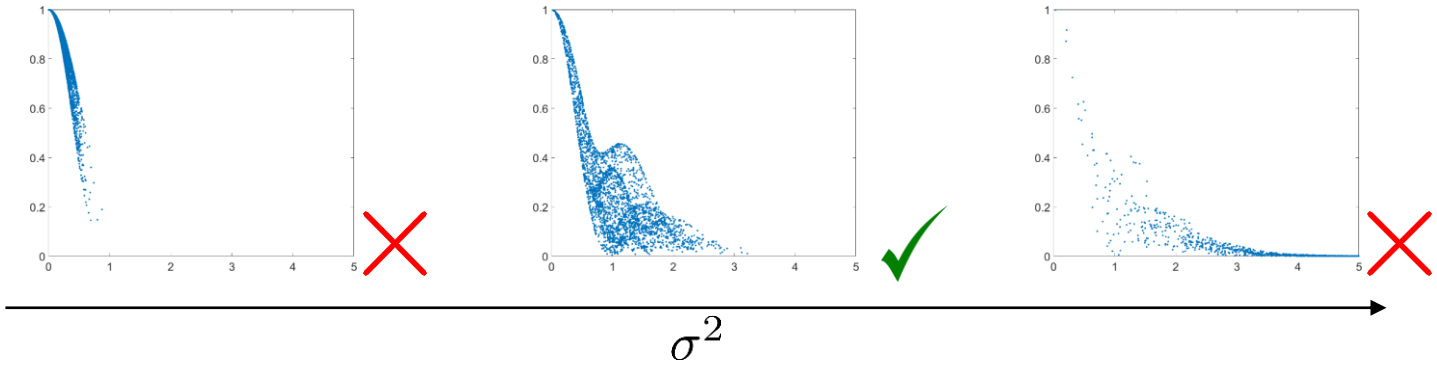

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Cross-validation

- *Can be very long !*
- *Used in practice [Sutherland2015]*

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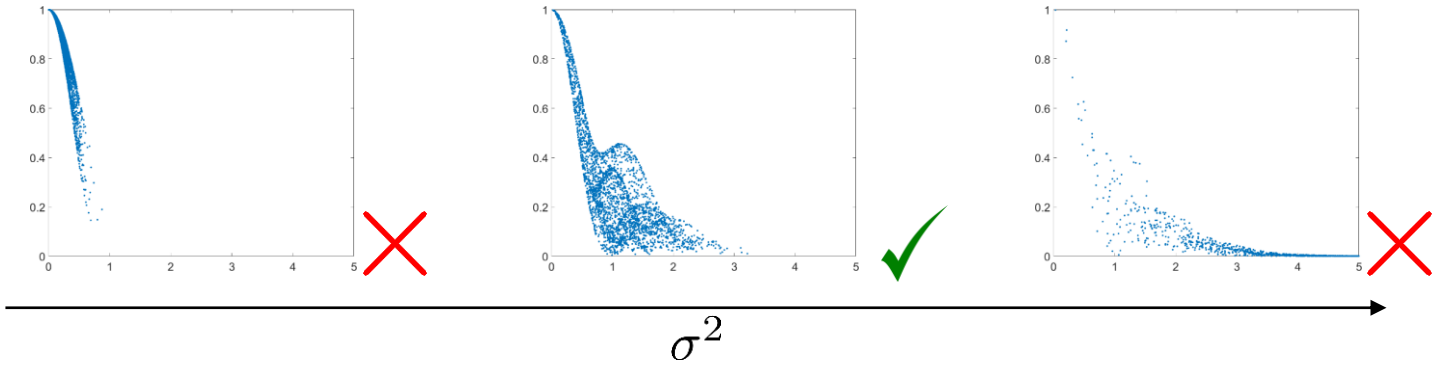

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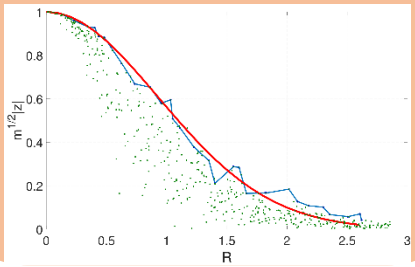
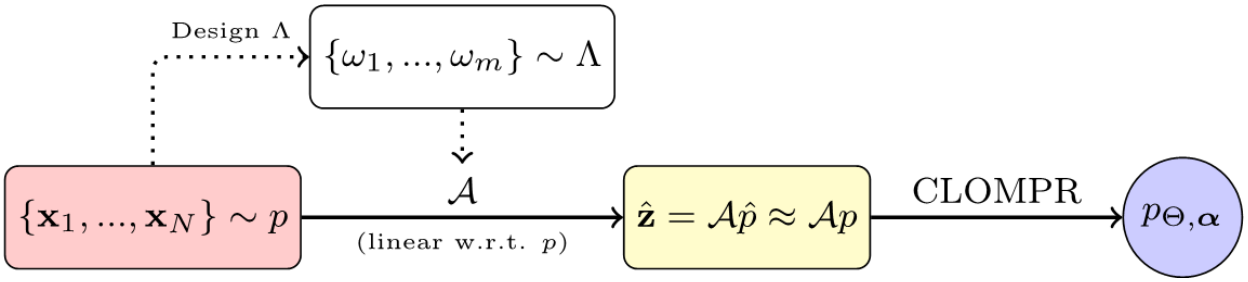
Proposed

Automatic

- *Partial pre-processing*
- *Heuristic based on GMMs-like distributions*



Summary



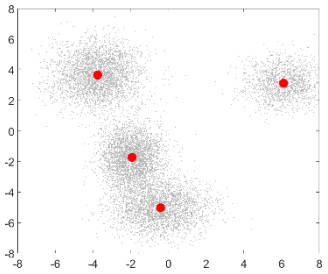
Given database, m , K

1. Design $\mathcal{A}$
   - Partial pre-processing to choose Λ
   - Draw $(\omega_1, ..., \omega_m) \overset{i.i.d.}{\sim} \Lambda$
2. Compute $\hat{\mathbf{z}} = \frac{1}{n}\left[\sum_i e^{-i\omega_j^T \mathbf{x}_i}\right]_{j=1}^{m}$
   - Online, distributed, GPU…
3. Derive mixture model $p_{\Theta,\alpha}$ with CLOMPR

Outline

Application : K-means, GMM

K-means



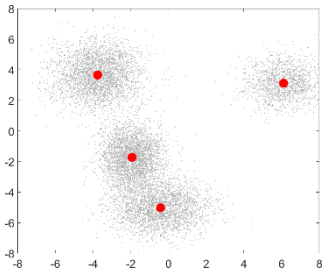
Classic approach

- Goal : $\min_\Theta \sum_{i=1}^{n} (\min_{1 \leq l \leq k} \|\mathbf{x}_i - \theta_l\|_2^2)$
- Algorithm : **Lloyd-Max** *[Lloyd 1982]*

(Matlab's `kmeans`*)*

Application : K-means, GMM

K-means



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- Algorithm : **Lloyd-Max** *[Lloyd 1982]*

(Matlab's `kmeans`*)*

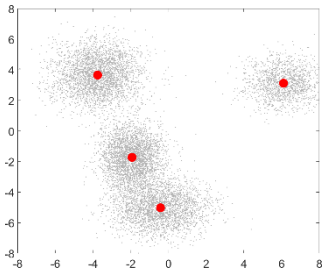
Compressive approach

- Model : $p_\theta = \delta_\theta \qquad \theta \in \mathbb{R}^d$

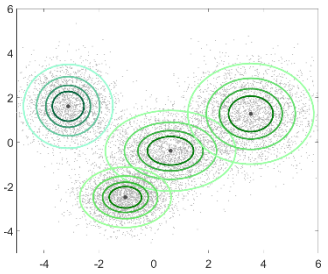
(clustered distribution = noisy mixture of Diracs)

Application : K-means, GMM

K-means



GMM diagonal cov.



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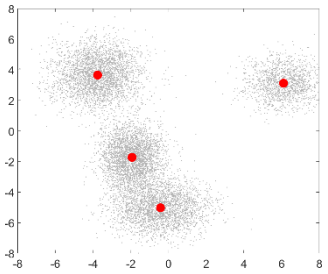

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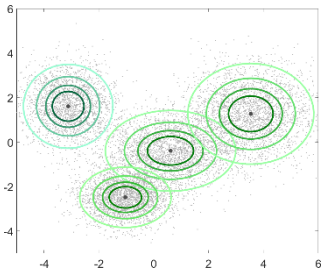

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Compressive approach

- Model : $p_\theta = \delta_\theta \qquad \theta \in \mathbb{R}^d$

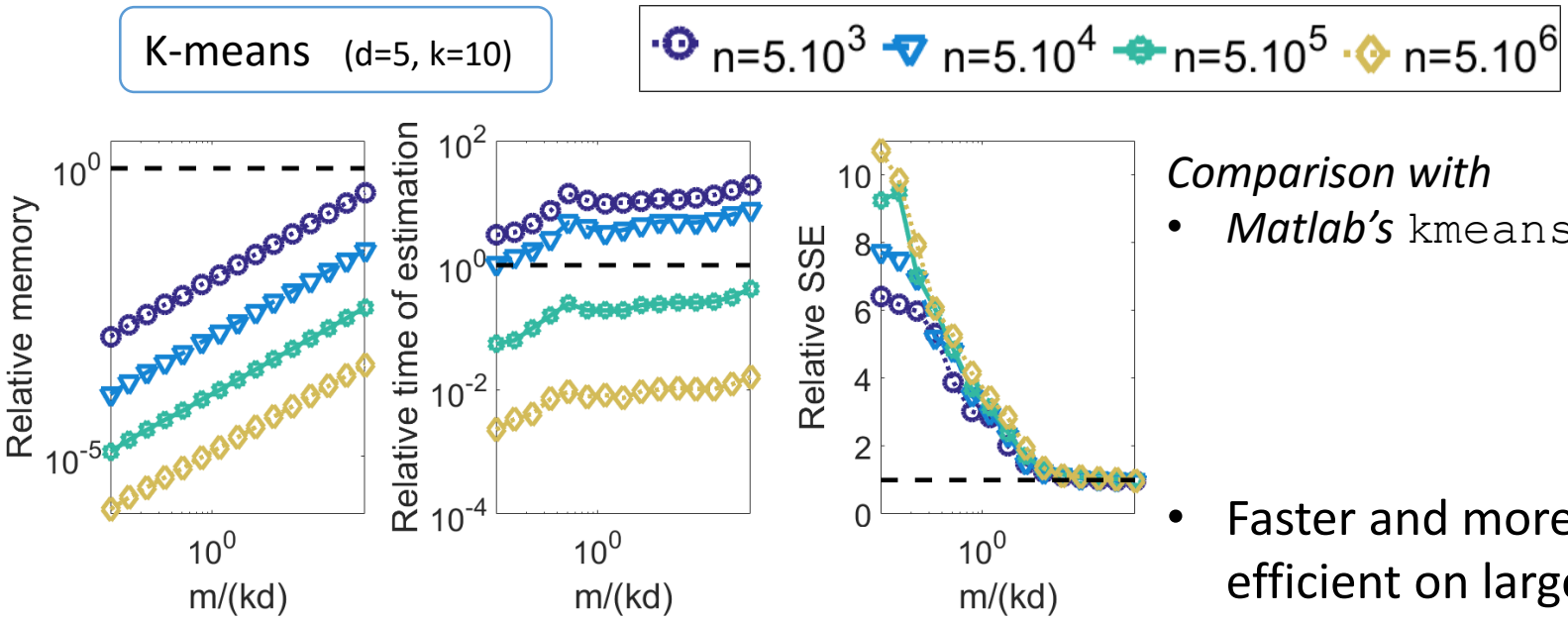
(clustered distribution = noisy mixture of Diracs)

Compressive approach

- Model : $\theta = (\mu, \sigma) \in \mathbb{R}^{2d}$
  $p_\theta = \mathcal{N}(\mu, diag(\sigma))$

Large-scale result

K-means (d=5, k=10)



Comparison with

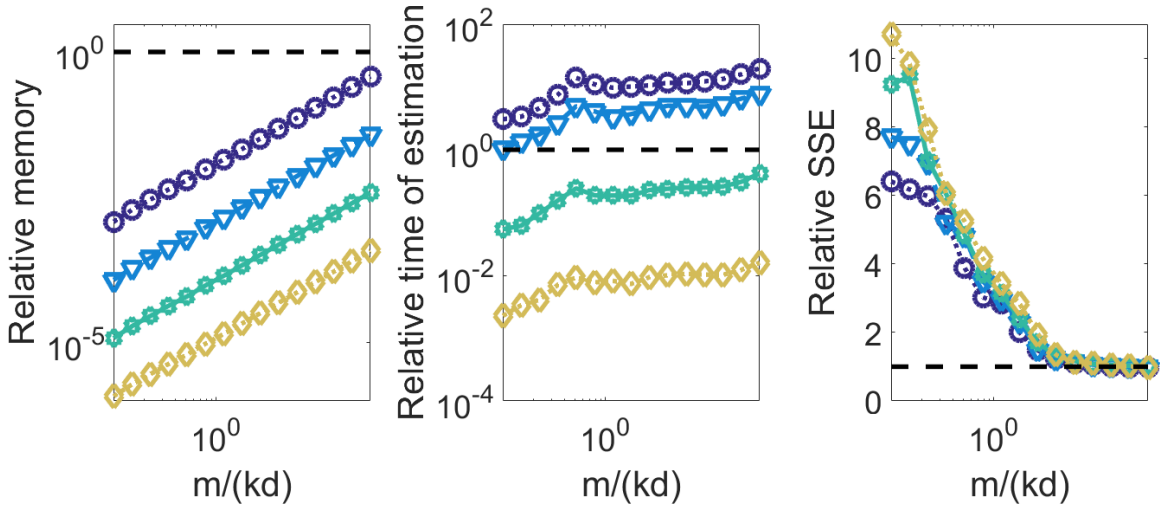
- *Matlab's* `kmeans`

- Faster and more memory efficient on large databases

- Number of measurements does not depend on N

Large-scale result

K-means (d=5, k=10)



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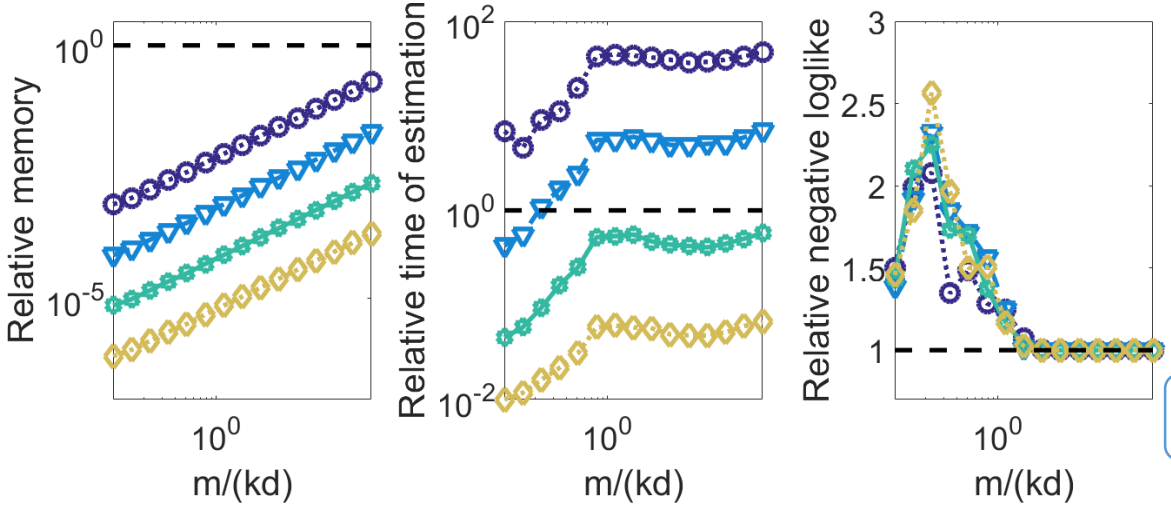
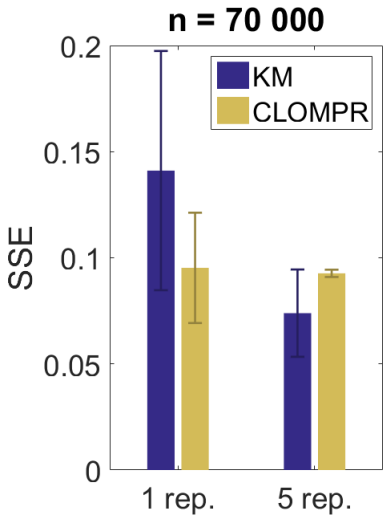
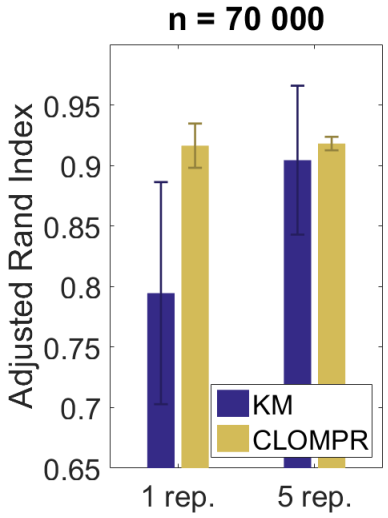


- Number of measurements does not depend on N

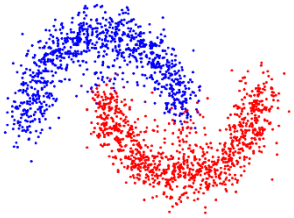
GMMs, diagonal cov. (d=5, k=5)

Application : spectral clustering *(Keriven, Tremblay, Traonmilin, Gribonval)*



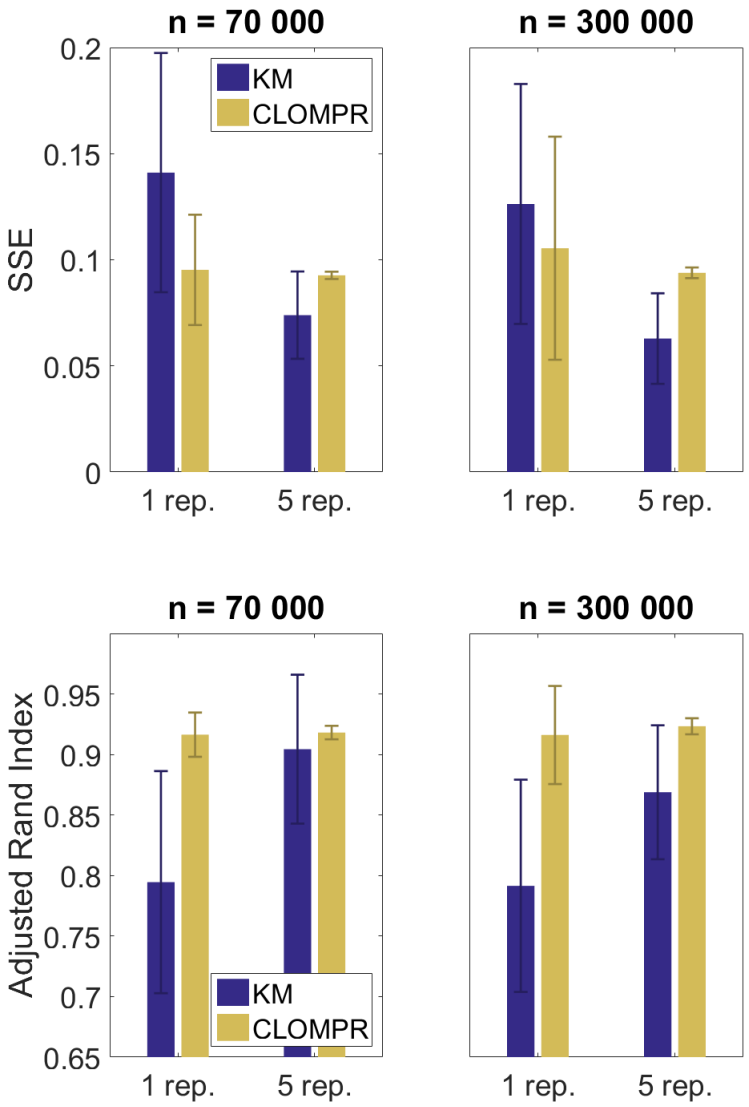


K-means (d=10, k=10, m=1000)
Mean and var. over 50 exp.



Spectral clustering for classification [Uw 2001], augmented MNIST database [Loosli 2007].

Application : spectral clustering *(Keriven, Tremblay, Traonmilin, Gribonval)*



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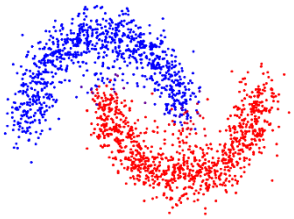

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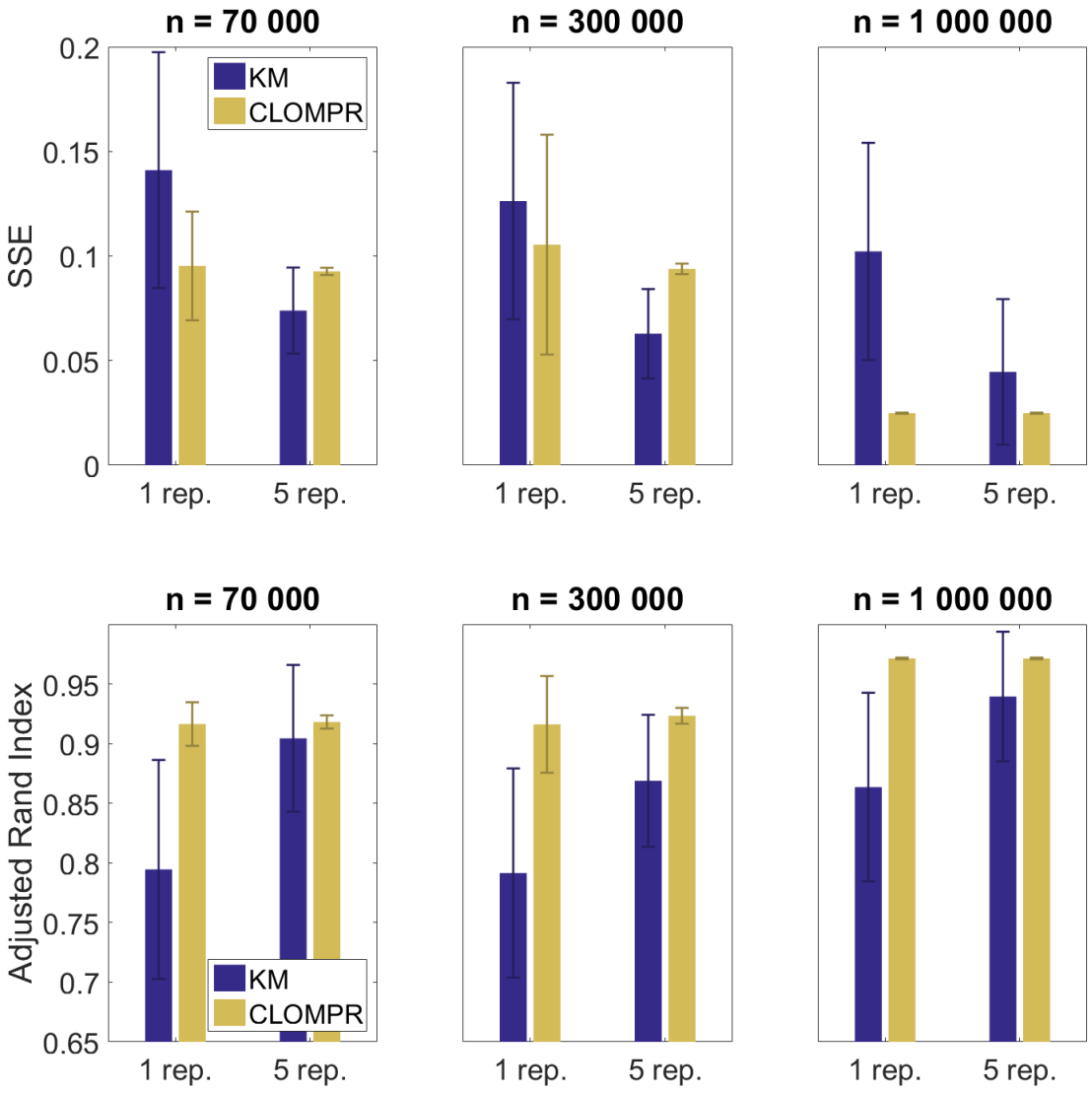

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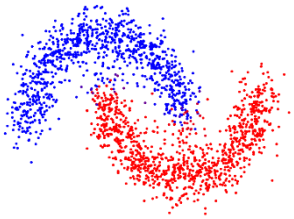

Spectral clustering for classification [Uw 2001], augmented MNIST database [Loosli 2007].

- CLOMPR **performs better** and is **more stable** with a large database

Application : speaker recognition

Variant of CLOMPR, faster at large k

	(Hierarchical) CLOMPR			EM
	$m = 10^3$	$m = 10^4$	$m = 10^5$	
$n = 3.10^5$	37.15	30.24	29.77	29.53
$\mathbf{n = 2.10^8}$	36.57	**28.96**	**28.59**	**N/A**

GMM (d=12, k=64)

Classical method for speaker recognition [Reynolds 2000] (for proof of concept) NIST 2005 database, MFCCs.

- Also performs better on a large database.

Outline

1. Introduction
2. Practical Approach
3. Results
4. **Theoretical analysis** *(Gribonval, Blanchard, Keriven, Traonmilin)*
5. Conclusion and outlooks

Information-preservation guarantees

Guarantee for CLOMPR ? Difficult ! (non-convex, random…)

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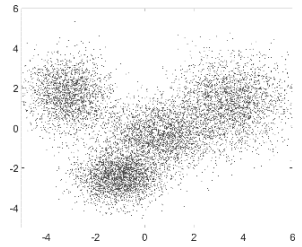
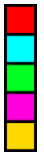
Guarantee for CLOMPR ? Difficult ! (non-convex, random…)



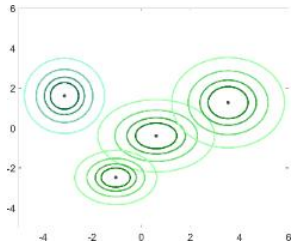
$\hat{p}$

$\mathcal{A}$



$\hat{\mathbf{z}} = \mathcal{A}\hat{p}$

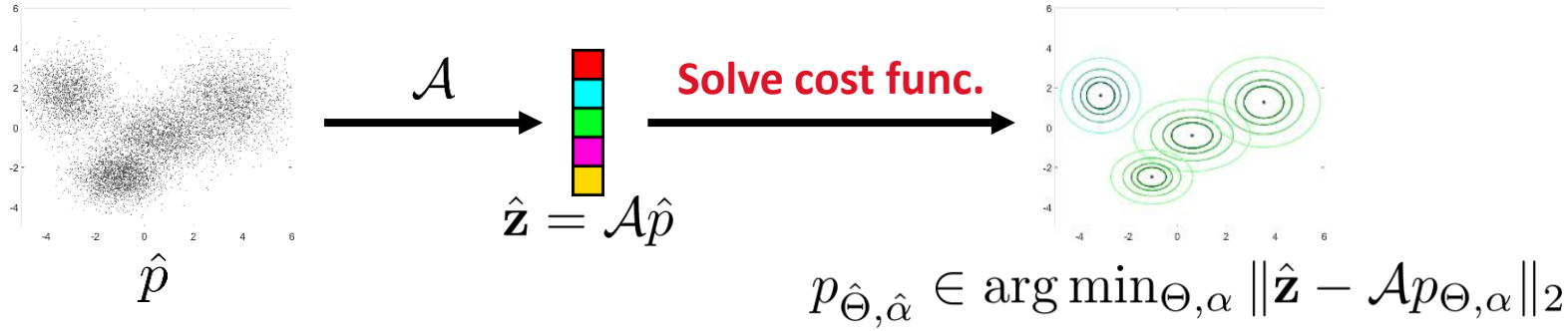
Solve cost func.



$$p_{\hat{\Theta},\hat{\alpha}} \in \arg\min_{\Theta,\alpha} \|\hat{\mathbf{z}} - \mathcal{A}p_{\Theta,\alpha}\|_2$$

Information-preservation guarantees

Guarantee for CLOMPR ? Difficult ! (non-convex, random…)



- Robustness to using $\hat{\mathbf{z}} = \mathcal{A}\hat{p}$ instead of $\mathbf{z} = \mathcal{A}p$?
- Robustness to p not being **exactly** a mixture model ?
- Guarantees in terms of usual learning cost functions ?
  - K-means : sum of distances to closest centroid
  - GMMs : negative log-likelihood

K-means: result

Goal minimize $R(\Theta) = \mathbb{E}_{\mathbf{x}\sim p^*}\left[\min_l \|\mathbf{x} - \theta_l\|_2^2\right]$ *(expected risk)*

K-means: result

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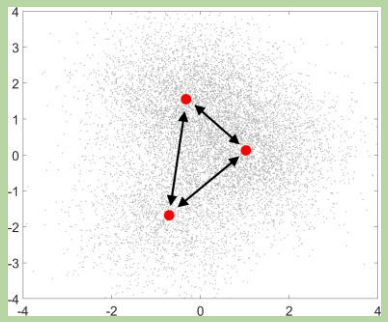
Hyp.
(on centroids, ***not samples****)*

- $\mathcal{E}$ - separation



- M - bounded domain
- **Reweighted** Fourier features
  (needed for theory, no effect in practice)

K-means: result

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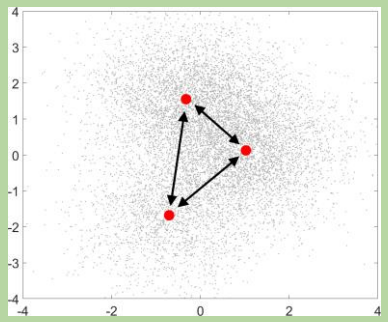

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  (needed for theory, no effect in practice)

If $m \geq \mathcal{O}\left(k^2 d^3 \texttt{polylog}(k,d)\log(M/\varepsilon)\right)$

w.h.p. $R(\hat{\Theta}) \lesssim R(\Theta^*) + \mathcal{O}\left(\sqrt{d^2 k/n}\right)$

Minimize cost func. (with hyp.)

Minimize expected risk (with hyp.)

GMMs with known covariance : result

Goal minimize $R(\Theta, \alpha) = \mathbb{E}_{\mathbf{x} \sim p^*} \Big[-\log p_{\Theta,\alpha}(\mathbf{x}) \Big]$ $p_{\Theta,\alpha} = \sum_l \alpha_l \mathcal{N}(\theta_l, \Sigma)$

(expected risk)

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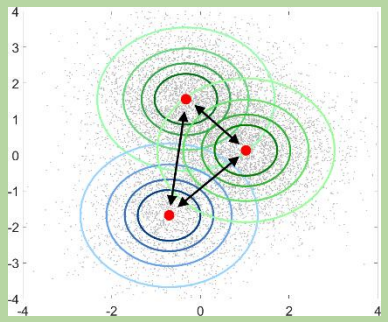
Hyp.
(on means, not samples)

- Large enough separation



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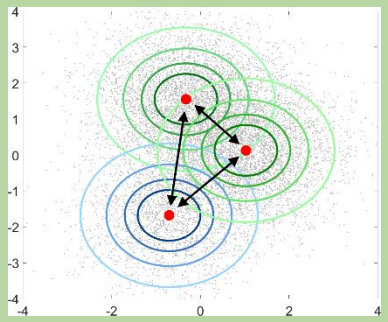


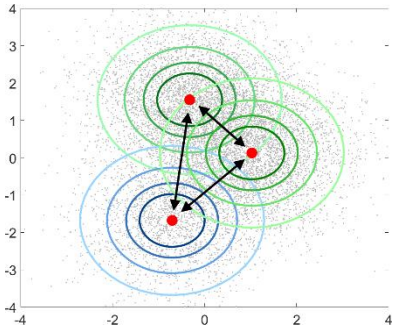
- M - bounded domain
- Fourier features

If m large enough w.h.p.

$$R(\hat{\Theta}, \hat{\alpha}) - R(\Theta^*, \alpha^*) \lesssim \inf_{\Theta,\alpha} \|p^* - p_{\Theta,\alpha}\|_{L^1} + \mathcal{O}\left(1/\sqrt{n}\right)$$

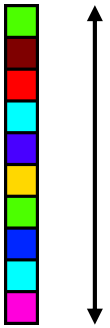
L1 distance from p* to the set of (separated) GMMs

GMM trade-off



Separation of means

Trade-off



Size of sketch

More high frequencies

Separation of means	Number of measurements
$\mathcal{O}\left(\sqrt{d \log k}\right)$	$m \geq \mathcal{O}\left(k^2 d^2 \cdot \texttt{polylog}(k,d)\right)$
$\mathcal{O}\left(\sqrt{d + \log k}\right)$	$m \geq \mathcal{O}\left(k^3 d^2 \cdot \texttt{polylog}(k,d)\right)$
$\mathcal{O}\left(\sqrt{\log k}\right)$	$m \geq \mathcal{O}\left(k^2 d^2 e^d \cdot \texttt{polylog}(k,d)\right)$

GMM with **unknown diagonal** covariance

$$\underbrace{\|p^* - p_{\hat{\Theta},\hat{\alpha}}\|}_{\text{Related to learning cost function ?}} \lesssim \inf_{\Theta,\alpha} \|p^* - p_{\Theta,\alpha}\|_{L^1} + \mathcal{O}\left(1/\sqrt{n}\right) + \underbrace{\mathcal{O}\left(1/\sqrt{m}\right)}_{\text{Efficiency of « compressive » approach ?}}$$

GMM with **unknown diagonal** covariance

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Nevertheless : $p_{\hat{\Theta},\hat{\alpha}} \xrightarrow{n,m\to\infty} p^*$ when p^* is exactly a GMM

Number of measurements

In theory, at least

$$m \geq \mathcal{O}(k^2 d^2)$$

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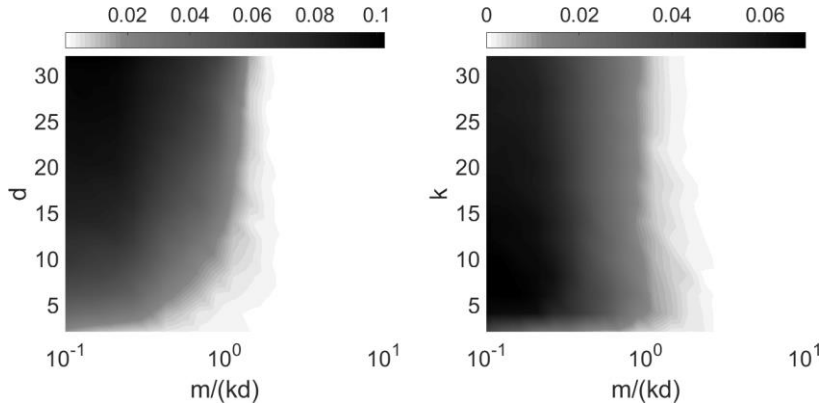
Empirically ?

$$m \approx \mathcal{O}(kd)$$

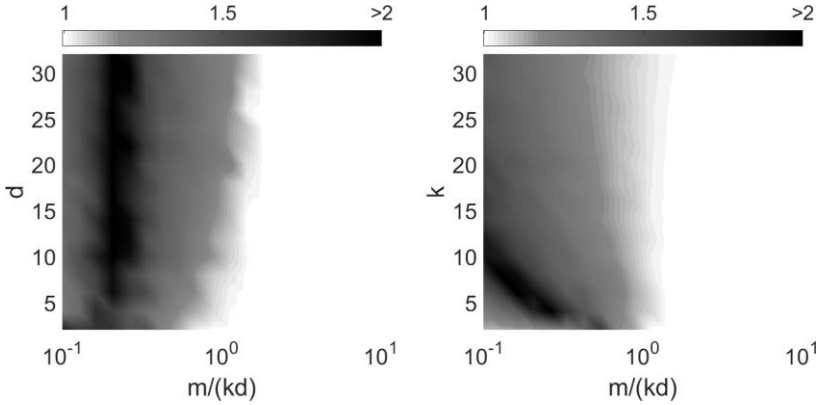
K-means

Relative SSE



GMMs, known cov.

Relative loglike



GMMs, diagonal cov.

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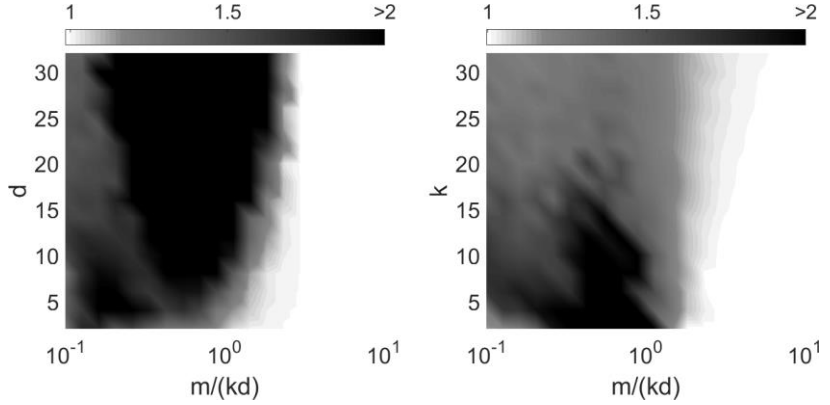
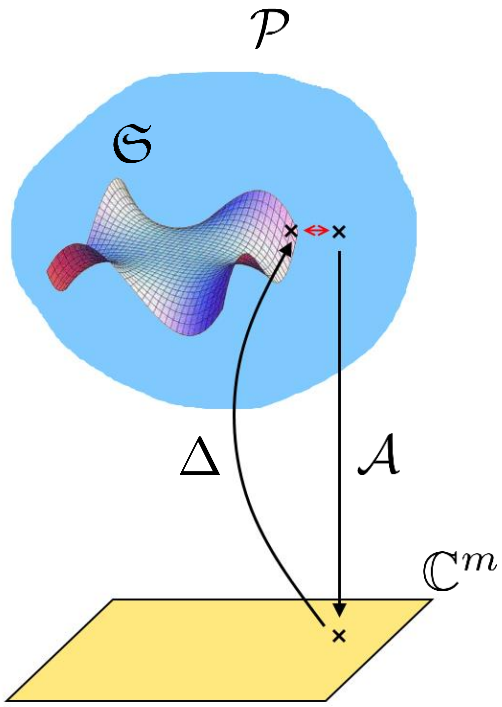


Sketch of proof : principle

Goal : Existence of instance Optimal Decoder

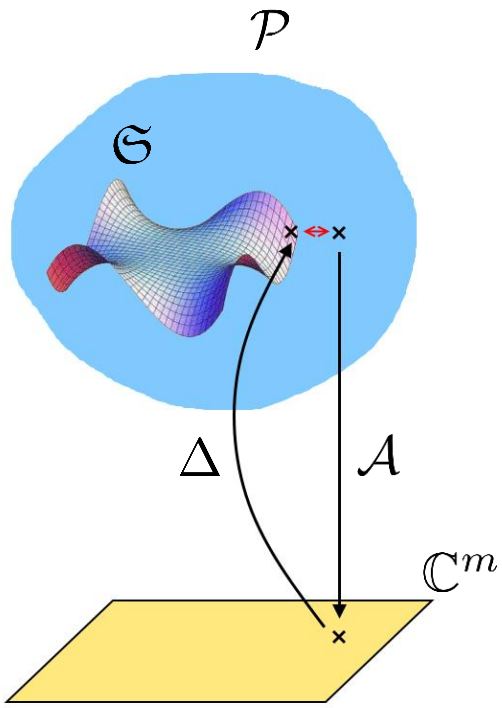


$$\|p^* - \Delta(\hat{\mathbf{z}})\| \lesssim d(p^*, \mathfrak{S}) + \underbrace{\|\mathcal{A}(p^* - \hat{p})\|}_{\mathcal{O}(1/\sqrt{n})}$$

Sketch of proof : principle

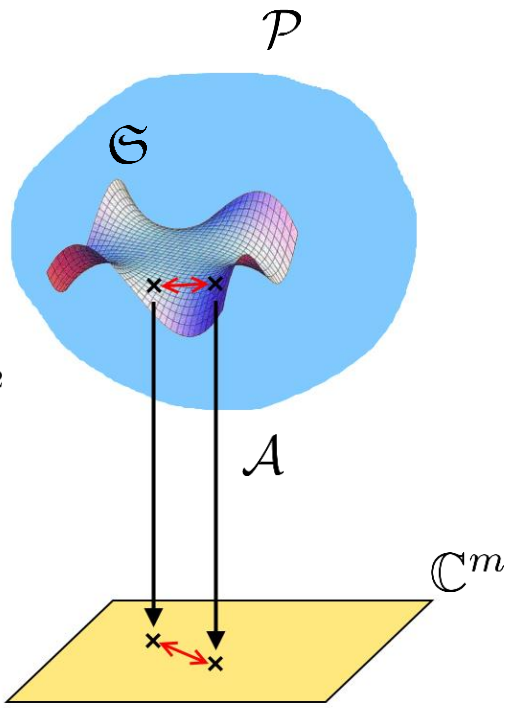
Goal : Existence of instance Optimal Decoder

Lower Restricted Isometry Property (LRIP)



[Bourrier 2014]

$\Delta(\mathbf{z}) = \arg\min_{p \in \mathfrak{S}} \|\mathbf{z} - \mathcal{A}p\|_2$



$\forall q, q' \in \mathfrak{S}$

$$\|p^* - \Delta(\hat{\mathbf{z}})\| \lesssim d(p^*, \mathfrak{S}) + \underbrace{\|\mathcal{A}(p^* - \hat{p})\|}_{\mathcal{O}(1/\sqrt{n})}$$

$$\|q - q'\| \lesssim \|\mathcal{A}(q - q')\|_2$$

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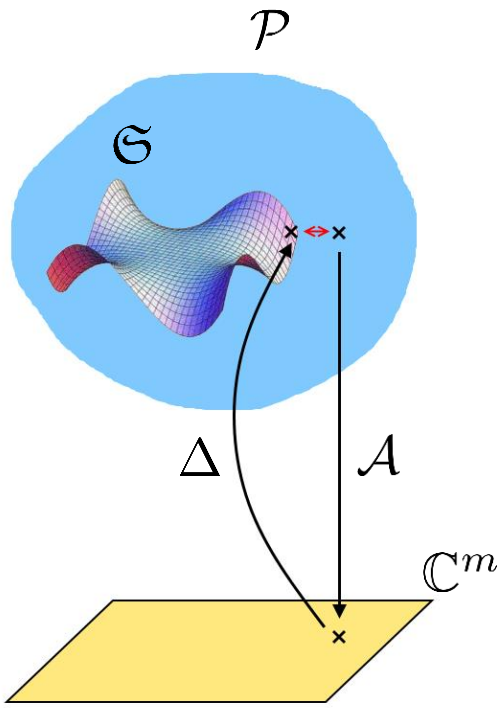

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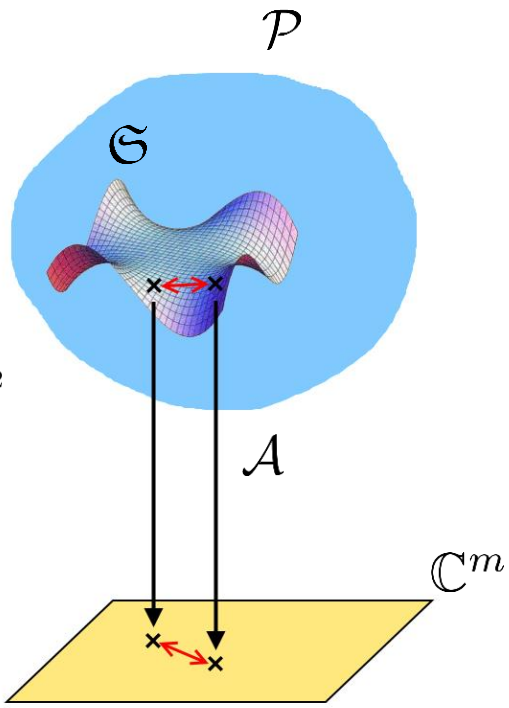

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$+\eta$

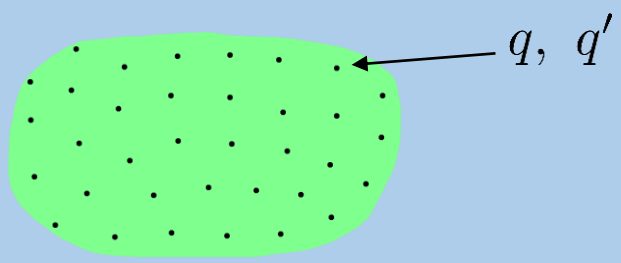
$$\|q - q'\| \lesssim \|\mathcal{A}(q - q')\|_2$$

$+\eta$

Ex : Quantization error

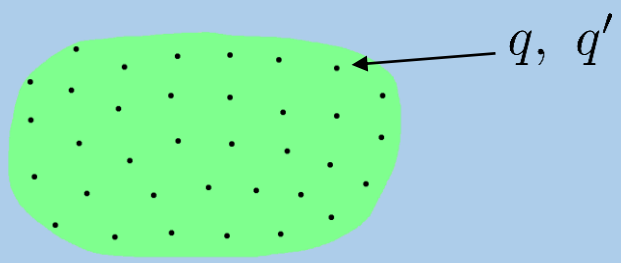
Proving the LRIP *(Gribonval, Blanchard, Keriven, Traonmilin)*

1 : Proving non-uniform LRIP



Proving the LRIP *(Gribonval, Blanchard, Keriven, Traonmilin)*

1 : Proving non-uniform LRIP



Kernel mean embedding *[Smola 2007]*
Random (Fourier) Features *[Rahimi 2007]*

$$\|\mathcal{A}(q-q')\|_2^2 \approx \|q-q'\|_\kappa^2$$

Hoeffding, Bernstein, chaining…

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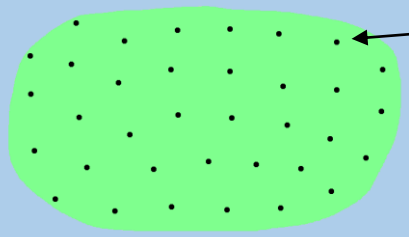

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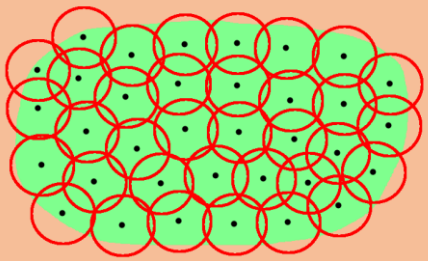
2 : Use $\mathcal{E}$ - coverings to extend to uniform LRIP



Proving the LRIP *(Gribonval, Blanchard, Keriven, Traonmilin)*

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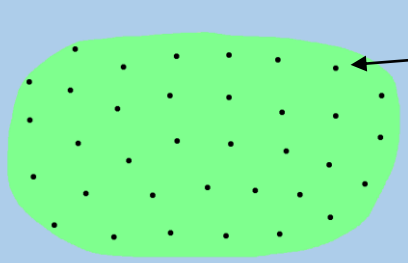

Kernel mean embedding *[Smola 2007]*
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$$\mathfrak{S} \text{ for } \eta > 0$$

Easy !

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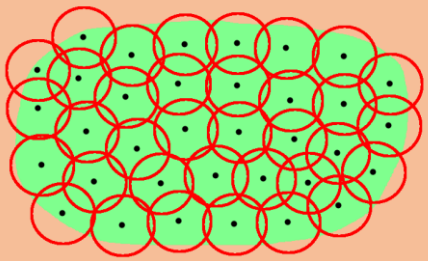

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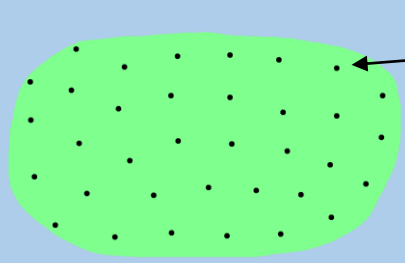

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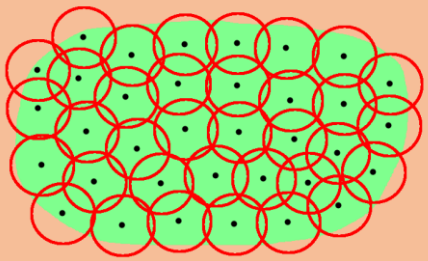

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Normalized Secant Set

$\left\{ \frac{q-q'}{\|q-q'\|_\kappa} \right\}$ for $\eta = 0$

- Finite-dimensional **Easy !**
- Infinite-dimensional **Difficult !**

Results

Sufficient Conditions

- $\mathfrak{S}$ has finite covering numbers

$$\eta = \mathcal{O}(1/\sqrt{m})$$

Not great !

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$$\eta = 0$$

Ex :

- *Mixtures of Diracs (K-means) with* $m \geq \mathcal{O}\left(k^2 d^3 \texttt{polylog}(k,d)\right)$
- *GMMs with known covariance*

Outline

1. Introduction
2. Practical Approach
3. Results
4. Theoretical analysis
5. **Conclusion and outlooks**

Conclusions

Contributions

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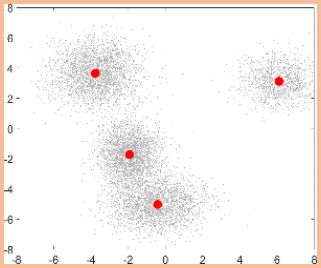
- **Information preservation guarantees** using infinite-dimensional Compressive Sensing

The SketchMLbox

SketchMLbox (`sketchml.gforge.inria.fr`)



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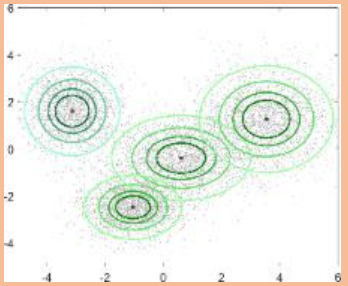
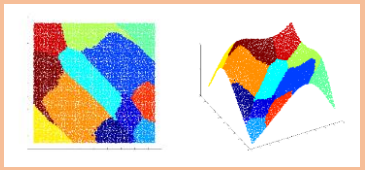


- GMMs with unknown diagonal covariance
- **Soon**:
  - Alpha-stable
  - Gaussian Locally Linear Mapping *[Deleforge 2014]*



- **Optimized for user-defined** $(\mathcal{A}p_\theta,\ \nabla_\theta \mathcal{A}p_\theta)$

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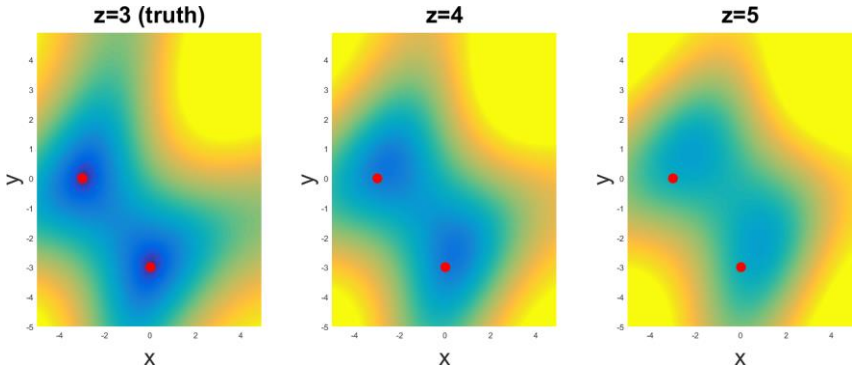

Cost function for a 3-GMM in dimension 1 at positions {x,y,z}={-3,0,3}

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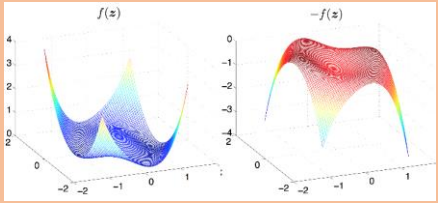

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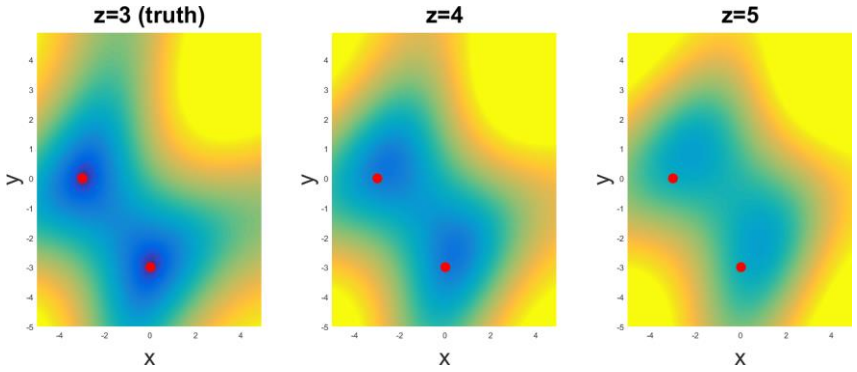

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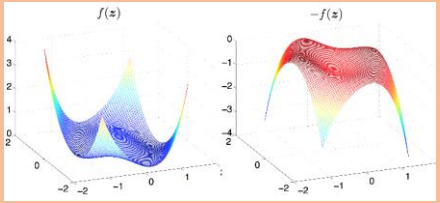

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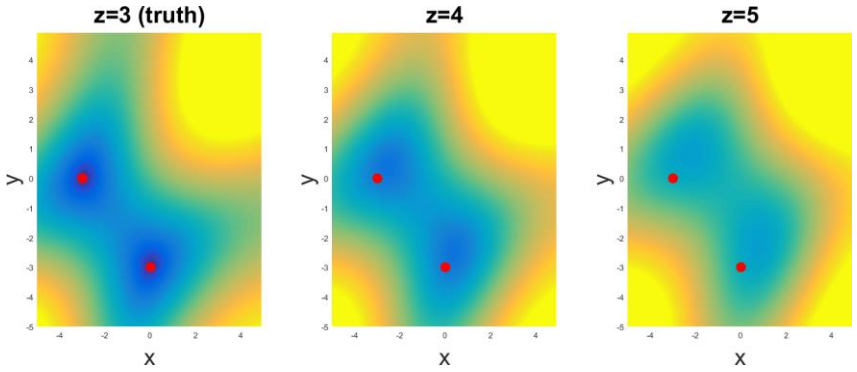

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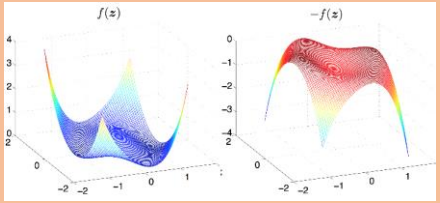

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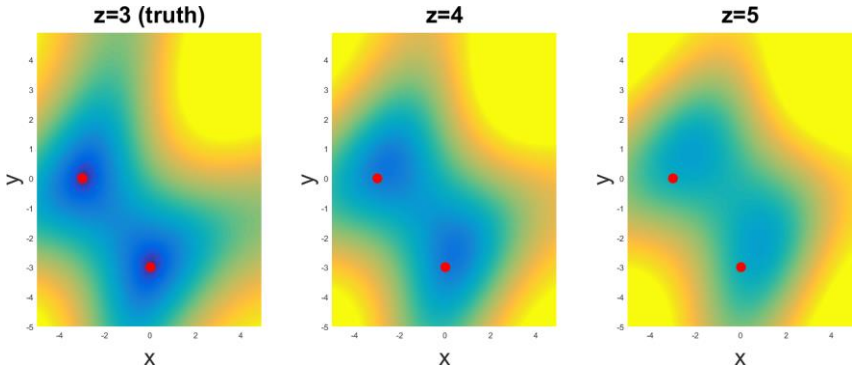

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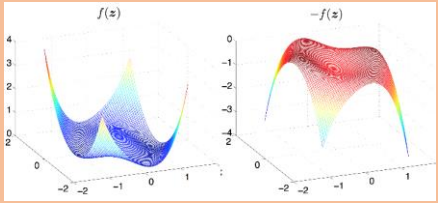
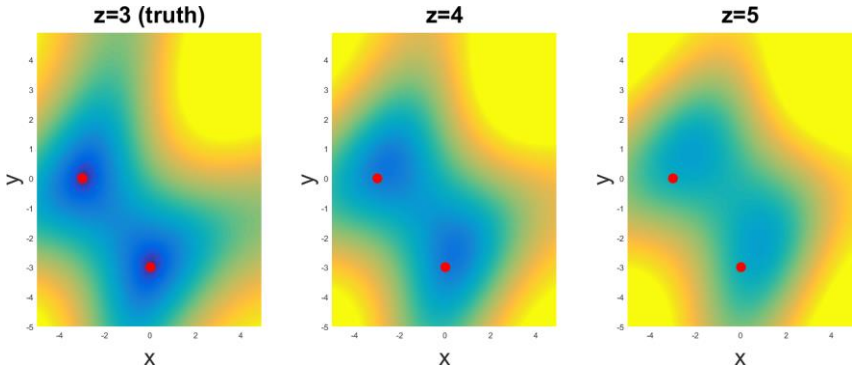


Recent result : locally block convex



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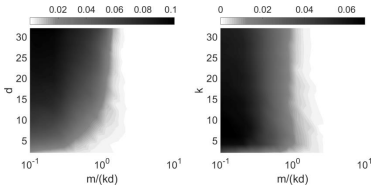

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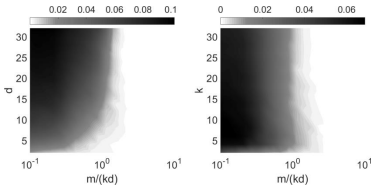

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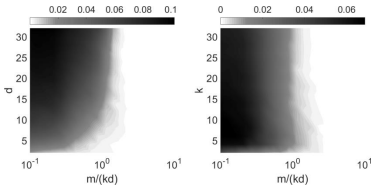

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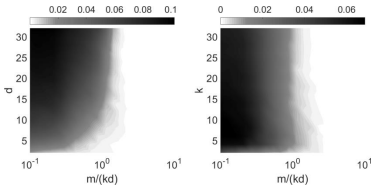

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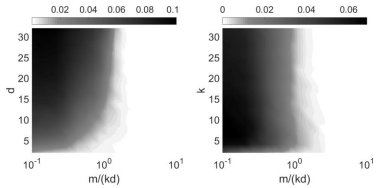

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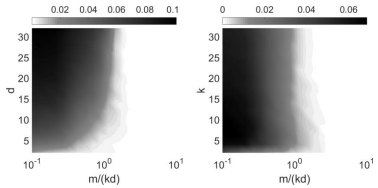

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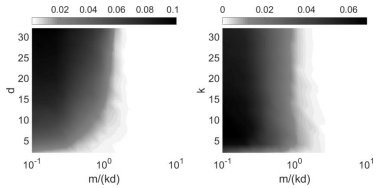

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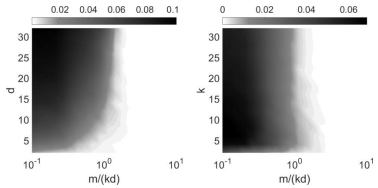

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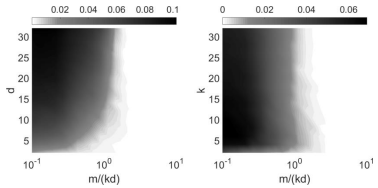

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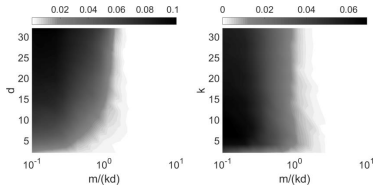

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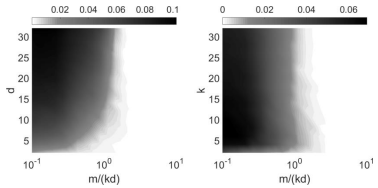

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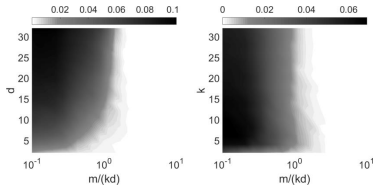

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   - CLOMPR and current sufficient conditions no longer valid…

Thank you !

- K., Bourrier, Gribonval, Perez. **Sketching for Large-Scale Learning of Mixture Models** *ICASSP 2016*
- K., Bourrier, Gribonval, Perez. **Sketching for Large-Scale Learning of Mixture Models** (extended version) *submitted to Information and Inference, arXiv:1606.0238*
- K., Tremblay, Gribonval, Traonmilin. **Compressive K-means** *ICASSP 2017*
- K., Tremblay, Gribonval. **SketchMLbox** (`sketchml.gforge.inria.fr`)
- Gribonval, Blanchard, K., Traonmilin. **Random moments for Sketched Statistical Learning** *submitted to AISTATS 2017, extended version soon*

Appendix : CLOMPR

Algorithm 2: Compressive mixture learning *à la* OMP: CLOMP ($T = K$) and CLOMPR ($T = 2K$)

Data: Empirical sketch $\hat{\mathbf{z}}$, sketching operator $\mathcal{A}$, sparsity K, number of iterations $T \geq K$
Result: Support Θ, weights $\boldsymbol{\alpha}$
$\hat{\mathbf{r}} \leftarrow \hat{\mathbf{z}}$; $\Theta \leftarrow \emptyset$;
for $t \leftarrow 1$ **to** T **do**
  Step 1: Find a normalized atom highly correlated with the residual with a gradient descent
    $\boldsymbol{\theta} \leftarrow \texttt{maximize}_{\boldsymbol{\theta}} \left(\mathrm{Re} \left\langle \frac{\mathcal{A}P_{\boldsymbol{\theta}}}{\|\mathcal{A}P_{\boldsymbol{\theta}}\|_2}, \hat{\mathbf{r}} \right\rangle_2 , \texttt{init} = \texttt{rand} \right)$;
  end
  Step 2: Expand support
    $\Theta \leftarrow \Theta \cup \{\boldsymbol{\theta}\}$;
  end
  Step 3: Enforce sparsity by Hard Thresholding if needed
    if $|\Theta| > K$ **then**
      $\boldsymbol{\beta} \leftarrow \arg\min_{\boldsymbol{\beta} \geq 0} \left\| \hat{\mathbf{z}} - \sum_{k=1}^{|\Theta|} \beta_k \frac{\mathcal{A}P_{\boldsymbol{\theta}_k}}{\|\mathcal{A}P_{\boldsymbol{\theta}_k}\|_2} \right\|_2$ Select K largest entries $\beta_{i_1}, ..., \beta_{i_K}$;
      Reduce the support $\Theta \leftarrow \{\boldsymbol{\theta}_{i_1}, ..., \boldsymbol{\theta}_{i_K}\}$;
    end
  end
  Step 4: Project to find weights
    $\boldsymbol{\alpha} \leftarrow \arg\min_{\boldsymbol{\alpha} \geq 0} \left\| \hat{\mathbf{z}} - \sum_{k=1}^{|\Theta|} \alpha_k \mathcal{A}P_{\boldsymbol{\theta}_k} \right\|_2$;
  end
  Step 5: Perform a gradient descent *initialized with current parameters*
    $\Theta, \boldsymbol{\alpha} \leftarrow \texttt{minimize}_{\Theta, \boldsymbol{\alpha}} \left(\left\| \hat{\mathbf{z}} - \sum_{k=1}^{|\Theta|} \alpha_k \mathcal{A}P_{\boldsymbol{\theta}_k} \right\|_2 , \texttt{init} = (\Theta, \boldsymbol{\alpha}) , \texttt{constraint} = \{\boldsymbol{\alpha} \geq 0\} \right)$;
  end
  Update residual: $\hat{\mathbf{r}} \leftarrow \hat{\mathbf{z}} - \sum_{k=1}^{|\Theta|} \alpha_k \mathcal{A}P_{\boldsymbol{\theta}_k}$
end
Normalize $\boldsymbol{\alpha}$ such that $\sum_{k=1}^{K} \alpha_k = 1$